# Still in the Mood: The Versatility of Subjunctive Markers in Modal Logic

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## 1 Introduction

In this paper we will discuss two linguistic frameworks for the formalization of modal discourse, to wit, standard modal logic with one or more actuality operators (cf. e.g. Hazen 1976, Crossley and Humberstone 1977, Peacocke 1978) and the "subjunctive" modal logic proposed by the first author (Wehmeier 2004, 2005). Both frameworks are motivated by certain expressive deficits of traditional quantified modal logic (cf. e.g. Hodes 1984, Wehmeier 2003, Kocurek 2015; for the propositional setting see Hazen 1978 as well as Hazen et al. 2013) that arise from the inability of unsupplemented modal languages to force the actual-world evaluation of material within the scope of a modal operator. The orthodox solution to this expressivity problem has been to endow modal languages with a scope-bearing actuality operator that is able, as it were, to shield the formulas it governs from the influence of any outlying modal operators. In Humberstone's (1982) memorable phrase, the actuality operator acts as an *inhibitor* to modal operators. Humberstone himself discusses (ibid.) a kind of dual to the actuality operator which he calls the subjunctive operator $\mathsf{S}$ and which accordingly acts as an *activator* relative to modal operators: Material outside the scope of the subjunctive operator is immune to the kind of evaluation-world relativization usually effected by governing modal operators, and invariably pertains to the actual world. While we believe that Humberstone's subjunctive operator deserves more detailed discussion than it has hitherto received, we will set it aside here to contrast actuality operator languages with modal languages in which the activating role of $\mathsf{S}$ is taken over by a subjunctive marker that attaches directly to sentence letters (in the propositional setting) or predicate symbols (in quantificational environments), as first proposed by Wehmeier (2004, 2005). While the actuality operator says to modal operators (paraphrasing Humberstone 1982 again), "Don't try to bind anything here, there's no world variable here", the subjunctive marker says, "Bind me—I'm a world variable!"

There has been some discussion recently over the relative capabilities and merits of actuality operator-based languages versus subjunctive marker-based languages (Humberstone 2004, Wehmeier 2012, French 2013, French 2015). This discussion has at times been hampered by misunderstandings of the subjunctive marker approach, caused in part by the lack of a comprehensive exposition of the framework. We thus set out in this paper to formulate in more explicit detail the subjunctive marker account of modal language, compare it (and some of its variants and extensions) to more orthodox languages containing actuality operators, and show not only that the subjunctive marker approach is every bit as versatile as the actuality operator approach, but also that it springs from a more unified and compelling syntactic and semantic picture.

In §2, we introduce both propositional and quantificational modal languages with one subjunctive marker. While the first author's original presentations of this approach had restricted the semantics, for simplicity, to Kripke models with universal accessibility relations, we here develop a general syntactic and semantic framework that does without this constraint. In parallel, we recall the associated modal languages with an actuality operator, and we show that, at the level of truth-at-a-world, modal languages of each type are translatable into the other.

In §3, we consider, on the one hand, propositional modal languages with infinitely many subjunctive markers and suitably indexed modal operators, and on the other hand, propositional modal languages with infinitely many actuality operators and indexed modal operators. Again we show that, as far as truth-at-a-world is concerned, languages of either type are translatable into the other.

Finally, in §4, we turn to the philosophical discussion of the results offered here and of various remarks concerning the relationship between actuality operator-based languages and subjunctive marker-based languages that can be found in the literature. The upshot of our discussion is that the versatility and internal coherence of the subjunctive marker approach gives it a leg up as a tool for the formalization of modal discourse.

## 2 Modal Logics with One Subjunctive Marker

In Wehmeier (2004, 2005) modal logic with subjunctive markers was defined relative to the class of Kripke models with universal accessibility relations. Since generalizations to models with non-universal accessibility relations due to the second author have occasionally been mentioned in the literature (e.g. Mackay 2013, Wehmeier 2013, and French 2015), it seems desirable to provide an official formulation of such systems in print.[1] Here we proceed to do so.

We focus first on propositional languages because some of the crucial conceptual innovations are most easily visible at this level. We will turn to a discussion of quantified modal languages in due course.

To begin with, we define a simple modal language with subjunctive markers.

**Definition 1** Let $\mathsf{Sent}$ be the countably infinite set of sentence letters, for which we write $p$, $q$, $r$ etc. The language $\mathcal{L}_s^1$ of subjunctive modal propositional logic with one subjunctive marker has as primitive symbols the members of $\mathsf{Sent}$, for each member $p$ of $\mathsf{Sent}$, a subjunctivized version $p^s$, the propositional connectives $\neg$ and $\wedge$, the modal operator $\Box$ and its subjunctivized version $\Box^s$, as well as opening and closing parentheses. The formulas of $\mathcal{L}_s^1$ are generated inductively as follows: Every sentence letter $p$ and every subjunctivized sentence letter $p^s$ is an $\mathcal{L}_s^1$-formula; whenever $\phi$ and $\psi$ are $\mathcal{L}_s^1$-formulas, so are $\neg\phi$, $(\phi \wedge \psi)$, $\Box\phi$ and $\Box^s\phi$.

$\mathcal{L}_s^1$ is interpreted in the familiar Kripke models $\mathbf{M} = (W, @, R, \mathcal{V})$, where

- $W$ is a set of indices, often called possible worlds,
- $@$ is a designated member of $W$, often called the *actual world* of $\mathbf{M}$,
- $R$ is a binary relation on $W$, and
- $\mathcal{V}$ is a family of functions $V_w : \mathsf{Sent} \to \{0, 1\}$ indexed by $W$.

The basic semantic notion of possible-world semantics is that of a formula's truth at a world, which we now define for $\mathcal{L}_s^1$.

**Definition 2** The following recursive clauses define the notion $\mathbf{M} \models_w \phi$ of an $\mathcal{L}_s^1$-formula's being $\mathcal{L}_s^1$-true at the world $w$ in the model $\mathbf{M}$.

- $\mathbf{M} \models_w p$ if and only if $V_@(p) = 1$ for $p \in \mathsf{Sent}$

[1]These generalizations were presented at the *Actuality and Subjunctivity* workshop held at UC Irvine in April of 2012.

- $\mathbf{M} \models_w p^s$ if and only if $V_w(p) = 1$ for $p \in \mathsf{Sent}$
- $\mathbf{M} \models_w \neg\phi$ if and only if $\mathbf{M} \not\models_w \phi$
- $\mathbf{M} \models_w (\phi \wedge \psi)$ if and only if $\mathbf{M} \models_w \phi$ and $\mathbf{M} \models_w \psi$
- $\mathbf{M} \models_w \Box\phi$ if and only if for every $v \in W$ such that $@Rv$, $\mathbf{M} \models_v \phi$
- $\mathbf{M} \models_w \Box^s\phi$ if and only if for every $v \in W$ such that $wRv$, $\mathbf{M} \models_v \phi$

In introducing the use of subjunctive markers, Wehmeier (2004, 2005) restricts attention to models without an accessibility relation (equivalently, with a universal accessibility relation) for reasons of simplicity. In such a setting, the use of two kinds of boxes, one indicative and one subjunctive, is unnecessary, and the indicative box alone suffices for full expressive power. This is because, in the clauses for the two boxes, we could just delete the conditions "such that $@Rv$" and "such that $wRv$", respectively, which would obliterate the distinction between them. The language $\mathcal{L}^1_s$, with the semantics given in definition 2, is the natural generalization of Wehmeier's original system to models with non-universal accessibility relations.

The system of propositional modal logic with subjunctive markers is expressively equivalent to the well known system of propositional modal logic with an actuality operator, whose definition we now recall.

**Definition 3** Let $\mathsf{Sent}$ be the set of sentence letters introduced in definition 1. The primitive symbols of the language $\mathcal{L}^1_{\mathsf{A}}$ of propositional modal logic with one actuality operator are the members of $\mathsf{Sent}$, the propositional connectives $\neg$ and $\wedge$, the necessity operator $\Box$, and the actuality operator $\mathsf{A}$, as well as opening and closing parentheses. Any sentence letter is an $\mathcal{L}^1_{\mathsf{A}}$-formula, and whenever $\phi$ and $\psi$ are $\mathcal{L}^1_{\mathsf{A}}$-formulas, so are $\neg\phi$, $(\phi \wedge \psi)$, $\Box\phi$, and $\mathsf{A}\phi$.

The language $\mathcal{L}^1_{\mathsf{A}}$ is interpreted in the same models as $\mathcal{L}^1_s$. Truth at a world is defined, for $\mathcal{L}^1_{\mathsf{A}}$-formulas, by the following recursive clauses.

**Definition 4** We write $\mathbf{M} \Vdash_w \phi$ to mean that the $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at the world $w$ of the model $\mathbf{M}$.

- $\mathbf{M} \Vdash_w p$ if and only if $V_w(p) = 1$
- $\mathbf{M} \Vdash_w \neg\phi$ if and only if $\mathbf{M} \nVdash_w \phi$
- $\mathbf{M} \Vdash_w (\phi \wedge \psi)$ if and only if $\mathbf{M} \Vdash_w \phi$ and $\mathbf{M} \Vdash_w \psi$

- $\mathbf{M} \Vdash_w \Box\phi$ if and only if for every $v \in W$ such that $wRv$, $\mathbf{M} \Vdash_v \phi$
- $\mathbf{M} \Vdash_w \mathsf{A}\phi$ if and only if $\mathbf{M} \Vdash_@ \phi$

The languages $\mathcal{L}_s^1$ and $\mathcal{L}_\mathsf{A}^1$ are expressively equivalent in the sense that one can map formulas of the one to formulas of the other in such a way that the input and output formulas are true at exactly the same worlds in exactly the same models. Call a sentence letter (respectively a box) *naked* if it does not carry a subjunctive marker superscript. Then to translate an $\mathcal{L}_s^1$-formula into an $\mathcal{L}_\mathsf{A}^1$-formula, one first prefixes every naked sentence letter and every naked box with an actuality operator and then erases every occurrence of the subjunctive marker. More precisely, we can define a translation function $t_{s,\mathsf{A}}$ from $\mathcal{L}_s^1$ to $\mathcal{L}_\mathsf{A}^1$ by means of the following clauses:

- $t_{s,\mathsf{A}}(p) = \mathsf{A}p$
- $t_{s,\mathsf{A}}(p^s) = p$
- $t_{s,\mathsf{A}}(\neg\phi) = \neg t_{s,\mathsf{A}}(\phi)$ and $t_{s,\mathsf{A}}(\phi \wedge \psi) = (t_{s,\mathsf{A}}(\phi) \wedge t_{s,\mathsf{A}}(\psi))$
- $t_{s,\mathsf{A}}(\Box\phi) = \mathsf{A}\Box t_{s,\mathsf{A}}(\phi)$
- $t_{s,\mathsf{A}}(\Box^s\phi) = \Box t_{s,\mathsf{A}}(\phi)$

Then we have:

**Lemma 1**

*For every $\mathcal{L}_s^1$-formula $\phi$, every model $\mathbf{M}$ and every world $w$ of $\mathbf{M}$,*

$$\mathbf{M} \models_w \phi \textit{ if and only if } \mathbf{M} \Vdash_w t_{s,\mathsf{A}}(\phi).$$

The proof proceeds by induction on $\phi$. Where $p$ is a naked sentence letter, $\mathbf{M} \models_w p$ if and only if $V_@(p) = 1$, which is the case if and only if $\mathbf{M} \Vdash_w \mathsf{A}p$, and $t_{s,\mathsf{A}}(p)$ just is $\mathsf{A}p$. For subjunctivized sentence letters $p^s$, we have that $\mathbf{M} \models_w p^s$ if and only if $V_w(p) = 1$, which is the case if and only if $\mathbf{M} \Vdash_w p$, and $t_{s,\mathsf{A}}(p^s)$ just is $p$. The cases for negation and conjunction follow immediately from the induction hypothesis. For the modal operators, consider first formulas of the form $\Box\phi$. We have $\mathbf{M} \models_w \Box\phi$ if and only if for every world $v$ with $@Rv$, $\mathbf{M} \models_v \phi$, which, by induction hypothesis, is the case if and only if for every world $v$ with $@Rv$, $\mathbf{M} \Vdash_v t_{s,\mathsf{A}}(\phi)$, which in turn holds just in case $\mathbf{M} \Vdash_w \mathsf{A}\Box t_{s,\mathsf{A}}(\phi)$, and $t_{s,\mathsf{A}}(\Box\phi)$ just is $\mathsf{A}\Box t_{s,\mathsf{A}}(\phi)$. Finally, $\mathbf{M} \models_w \Box^s\phi$ if and only if for every world $v$ with $wRv$, $\mathbf{M} \models_v \phi$, which in turn holds (by induction hypothesis) just in case for every $v$ with $wRv$, $\mathbf{M} \Vdash_v t_{s,\mathsf{A}}(\phi)$, and that means that $\mathbf{M} \Vdash_w \Box t_{s,\mathsf{A}}(\phi)$. But $t_{s,\mathsf{A}}(\Box^s \phi)$ just is $\Box t_{s,\mathsf{A}}(\phi)$, QED.

Toward a translation in the other direction, call an occurrence in an $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$ of a sentence letter (respectively, of a box) A-*independent* if it does not lie within the direct scope of an actuality operator in $\phi$.[2] Then, to translate an $\mathcal{L}^1_{\mathsf{A}}$-formula $\psi$ into an $\mathcal{L}^1_s$-formula $t_{\mathsf{A},s}(\psi)$, one attaches a subjunctive marker superscript to every A-independent sentence letter and box, and then erases every occurrence of A.

In order to give a more precise definition of the translation function $t_{\mathsf{A},s}$, we first define an auxiliary mapping act from $\mathcal{L}^1_s$ to $\mathcal{L}^1_s$ as follows:

- $\mathsf{act}(p) = \mathsf{act}(p^s) = p$
- $\mathsf{act}(\neg\phi) = \neg\mathsf{act}(\phi)$ and $\mathsf{act}(\phi \wedge \psi) = \mathsf{act}(\phi) \wedge \mathsf{act}(\psi)$
- $\mathsf{act}(\Box\phi) = \mathsf{act}(\Box^s\phi) = \Box\phi$

The translation $t_{\mathsf{A},s}$ from $\mathcal{L}^1_{\mathsf{A}}$ to $\mathcal{L}^1_s$ can then be defined by the following recursive clauses:

- $t_{\mathsf{A},s}(p) = p^s$
- $t_{\mathsf{A},s}(\neg\phi) = \neg t_{\mathsf{A},s}(\phi)$ and $t_{\mathsf{A},s}(\phi \wedge \psi) = t_{\mathsf{A},s}(\phi) \wedge t_{\mathsf{A},s}(\psi)$
- $t_{\mathsf{A},s}(\Box\phi) = \Box^s t_{\mathsf{A},s}(\phi)$
- $t_{\mathsf{A},s}(\mathsf{A}\phi) = \mathsf{act}(t_{\mathsf{A},s}(\phi))$.

The following lemma shows that act simulates the actuality operator within $\mathcal{L}^1_s$.

**Lemma 2**

*For every $\mathcal{L}^1_s$-formula $\phi$, every model $\mathbf{M}$ and every world $w$ of $\mathbf{M}$,*

$$\mathbf{M} \models_w \mathsf{act}(\phi) \text{ if and only if } \mathbf{M} \models_@ \phi.$$

[2] A symbol occurrence $o$ lies within the direct scope of an actuality operator in $\phi$ just in case there is a subformula occurrence $\mathsf{A}\psi$ in $\phi$ such that $o$ lies in $\psi$ but not within the scope of a box in $\psi$. In other words, $o$ lies within the direct scope of A in $\phi$ just in case $o$ lies within the scope of an occurrence of A, but not within the scope of any occurrence of the box that itself lies within the scope of the original A-occurrence.

**Proof.** We proceed by induction on $\phi$.

When $\phi$ is a naked sentence letter $p$, $\mathsf{act}(\phi)$ is just $p$, so $\mathbf{M} \models_w \mathsf{act}(\phi)$ iff $\mathbf{M} \models_w p$ iff $V_@(p) = 1$ iff $\mathbf{M} \models_@ p$.

When $\phi$ is a subjunctivized sentence letter $p^s$, $\mathsf{act}(\phi)$ is still $p$, so $\mathbf{M} \models_w \mathsf{act}(\phi)$ iff $\mathbf{M} \models_w p$ iff $V_@(p) = 1$ iff $\mathbf{M} \models_@ p^s$.

The cases of negation and conjunction follow immediately from the induction hypothesis.

If $\phi$ is $\Box\psi$, $\mathsf{act}(\phi)$ is just $\phi$ itself, so $\mathbf{M} \models_w \mathsf{act}(\phi)$ iff $\mathbf{M} \models_w \Box\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models_v \psi$ iff $\mathbf{M} \models_@ \Box\psi$.

Finally, if $\phi$ is $\Box^s\psi$, $\mathsf{act}(\phi)$ is $\Box\psi$, so $\mathbf{M} \models_w \mathsf{act}(\phi)$ iff $\mathbf{M} \models_w \Box\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models_v \psi$ iff $\mathbf{M} \models_@ \Box^s\psi$. QED.

We can now prove a translation result in the direction from $\mathcal{L}^1_{\mathsf{A}}$ to $\mathcal{L}^1_s$:

**Lemma 3**

*For every $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$, every model $\mathbf{M}$ and every world $w$ of $\mathbf{M}$,*

$$\mathbf{M} \Vdash_w \phi \text{ if and only if } \mathbf{M} \models_w t_{\mathsf{A},s}(\phi).$$

The proof proceeds by induction on $\phi$.

If $\phi$ is a sentence letter $p$, $t_{\mathsf{A},s}(\phi)$ is its subjunctivized version $p^s$, and by definition we have $\mathbf{M} \Vdash_w p$ iff $V_w(p) = 1$ iff $\mathbf{M} \models_w p^s$ iff $\mathbf{M} \models_w t_{\mathsf{A},s}(\phi)$.

The cases of negations and conjunctions follow immediately from the induction hypothesis.

If $\phi$ is a necessitation $\Box\psi$, $t_{\mathsf{A},s}(\phi)$ is $\Box^s t_{\mathsf{A},s}(\psi)$. We then have $\mathbf{M} \Vdash_w \phi$ iff $\mathbf{M} \Vdash_w \Box\psi$ iff for all $v$ with $wRv$, $\mathbf{M} \Vdash_v \psi$ iff (by induction hypothesis) for all $v$ with $wRv$, $\mathbf{M} \models_v t_{\mathsf{A},s}(\psi)$ iff $\mathbf{M} \models_w \Box^s t_{\mathsf{A},s}(\psi)$.

If $\phi$ is $\mathsf{A}\psi$, $t_{\mathsf{A},s}(\phi)$ is $\mathsf{act}(t_{\mathsf{A},s}(\psi))$. Thus we have $\mathbf{M} \Vdash_w \phi$ iff $\mathbf{M} \Vdash_w \mathsf{A}\psi$ iff $\mathbf{M} \Vdash_@ \psi$ iff (by induction hypothesis) $\mathbf{M} \models_@ t_{\mathsf{A},s}(\psi)$ iff (by the previous lemma) $\mathbf{M} \models_w \mathsf{act}(t_{\mathsf{A},s}(\phi))$. QED.

Call an $\mathcal{L}^1_s$-formula $\phi$ *subjunctively closed* if every occurrence of a subjunctivized sentence letter and every occurrence of a subjunctivized box in $\phi$ lies within the scope of some box.[3] It is easy to see that a subjunctively closed $\mathcal{L}^1_s$-formula is either true at all worlds of a model or at none; in other words the truth

[3] It does not matter whether we require every such occurrence to lie within the scope of a naked box, or just within the scope of some box, whether naked or subjunctivized. This is because on either version of the definition subjunctivized boxes themselves must ultimately lie within the scope of a naked box to achieve subjunctive closure, and so all the material within the scope of a subjunctive box will also lie within the scope of a naked box.

value of a subjunctively closed formula is independent of the world of evaluation. We may thus define a subjunctively closed $\mathcal{L}^1_s$-formula $\phi$ to be ($\mathcal{L}^1_s$-) *true in a model* $\mathbf{M}$, $\mathbf{M} \models \phi$, just in case it is $\mathcal{L}^1_s$-true at all worlds (equivalently, at some world or other; equivalently, at the actual world) of $\mathbf{M}$. On the $\mathcal{L}^1_{\mathsf{A}}$-side, it is standard to define an arbitrary formula $\phi$ to be ($\mathcal{L}^1_{\mathsf{A}}$-) true in a model $\mathbf{M}$, $\mathbf{M} \Vdash \phi$, just in case $\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at the actual world of $\mathbf{M}$, i.e. $\mathbf{M} \Vdash \phi :\Leftrightarrow \mathbf{M} \Vdash_{@} \phi$ (this notion of truth for $\mathcal{L}^1_{\mathsf{A}}$ is usually called *real-world truth*).

Noting that the values of the function $\mathsf{act}$ are always subjunctively closed formulas, we therefore obtain the following translation result.

**Corollary 1**

*For every $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$ and every model $\mathbf{M}$,*

$$\mathbf{M} \Vdash \phi \textit{ if and only if } \mathbf{M} \models \mathsf{act}(t_{\mathsf{A},s}(\phi)).$$

The proof is obvious: $\mathbf{M} \Vdash \phi$ if and only if $\mathbf{M} \Vdash_{@} \phi$, and by Lemma 3, we have $\mathbf{M} \Vdash_{@} \phi$ if and only if $\mathbf{M} \models_{@} t_{\mathsf{A},s}(\phi)$. By Lemma 2, for any $w$, $\mathbf{M} \models_{@} t_{\mathsf{A},s}(\phi)$ if and only if $\mathbf{M} \models_w \mathsf{act}(t_{\mathsf{A},s}(\phi))$. Since $\mathsf{act}(t_{\mathsf{A},s}(\phi))$ is subjunctively closed, the latter is equivalent, for any $w$, to $\mathbf{M} \models \mathsf{act}(t_{\mathsf{A},s}(\phi))$. QED.

Corollary 1 establishes that the function $\mathsf{act} \circ t_{\mathsf{A},s}$ is a translation from $\mathcal{L}^1_{\mathsf{A}}$ with real-world truth into the subjunctively closed fragment of $\mathcal{L}^1_s$ with $\mathcal{L}^1_s$-truth *tout court*. This generalizes a corresponding result mentioned without proof in Wehmeier (2004). It is perhaps instructive to put our translation $\mathsf{act} \circ t_{\mathsf{A},s}$ alongside (a generalization[4] of) the one given by French (2013, p. 1692) which likewise translates $\mathcal{L}^1_{\mathsf{A}}$ with real-world truth into the subjunctively closed fragment of $\mathcal{L}^1_s$.

French defines two translation functions, $\tau_1$ and $\tau_S$, which recursively call each other, as follows. (Note that the values of $\tau_1$ are always subjunctively closed formulas.)

$$\begin{array}{ll}
\tau_1(p) = p & \tau_S(p) = p^s \\
\tau_1(\neg\phi) = \neg\tau_1(\phi) & \tau_S(\neg\phi) = \neg\tau_S(\phi) \\
\tau_1(\phi \wedge \psi) = (\tau_1(\phi) \wedge \tau_1(\psi)) & \tau_S(\phi \wedge \psi) = (\tau_S(\phi) \wedge \tau_S(\psi)) \\
\tau_1(\Box\phi) = \Box\tau_S(\phi) & \tau_S(\Box\phi) = \Box^s\tau_S(\phi) \\
\tau_1(\mathsf{A}\phi) = \tau_1(\phi) & \tau_S(\mathsf{A}\phi) = \tau_1(\phi)
\end{array}$$

We can think of $\tau_1$ as giving the appropriate translation of an $\mathcal{L}^1_{\mathsf{A}}$-formula when the world of evaluation is set to the actual world, and of $\tau_S$ as giving the appropriate translation when the world of evaluation is arbitrary: The $\mathcal{L}^1_{\mathsf{A}}$-truth value of

[4]The generalization concerns the translation of subjunctivized boxes, which are absent from the modal languages considered by Wehmeier (2004) and French (2013).

the formula $p$ at the actual world is the same as the $\mathcal{L}^1_s$-truth value (*simpliciter*) of $p$, hence $\tau_1(p) = p$. By contrast, the $\mathcal{L}^1_{\mathsf{A}}$-truth value of $p$ at some arbitrary world $w$ is the same as the $\mathcal{L}^1_s$-truth value of $p^s$ at $w$, hence $\tau_S(p) = p^s$. Ignoring the trivial cases of the truth-functional connectives, we next observe that the $\mathcal{L}^1_{\mathsf{A}}$-truth value of $\Box\phi$ at the actual world is 1 if and only if for each world $v$ accessible from the actual world, the $\mathcal{L}^1_{\mathsf{A}}$-truth value of $\phi$ at $v$ is 1. So if $\tau_S(\phi)$ is an $\mathcal{L}^1_s$-formula that is $\mathcal{L}^1_s$-true at any $v$ just in case $\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at $v$, this is equivalent to the $\mathcal{L}^1_s$-truth value (*simpliciter*) of $\Box\tau_S(\phi)$ being 1; hence $\tau_1(\Box\phi) = \Box\tau_S(\phi)$. When the world of evaluation is arbitrary, the only thing that changes is that we must consider the worlds accessible from the world of evaluation rather than those accessible from the actual world; hence $\tau_S(\Box\phi) = \Box^s\tau_S(\phi)$. Finally, $\mathsf{A}\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at the actual world if and only if $\phi$ is true at the actual world, so we can set $\tau_1(\mathsf{A}\phi)$ equal to $\tau_1(\phi)$; similarly, $\mathsf{A}\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at an arbitrary world $w$ just in case $\phi$ is $\mathcal{L}^1_{\mathsf{A}}$-true at the actual world, which we know to be the case if and only if $\tau_1(\phi)$ is $\mathcal{L}^1_s$-true *simpliciter*, and by subjunctive closure this is the case if and only if $\tau_1(\phi)$ is $\mathcal{L}^1_s$-true at $w$.

Thus, as French shows, we have that $\mathbf{M} \Vdash \phi$ if and only if $\mathbf{M} \models \tau_1(\phi)$ and that $\mathbf{M} \Vdash_w \phi$ if and only if $\mathbf{M} \models_w \tau_S(\phi)$, for any world $w$ of $\mathbf{M}$. So $\tau_1$ accomplishes the same as our translation $\mathsf{act} \circ t_{\mathsf{A},s}$, and $\tau_S$ accomplishes the same as our $t_{\mathsf{A},s}$. But while French's $\tau_1$ and $\tau_S$ recursively call each other, our $t_{\mathsf{A},s}$ is defined autonomously ($\mathsf{act} \circ t_{\mathsf{A},s}$ obviously does call $t_{\mathsf{A},s}$). This simplification, if it is one, in our translation is possible because calling $\tau_1$ in the clause for $\tau_S(\mathsf{A}\phi)$ turns out to be unnecessary, given the availability of $\mathsf{act}$: It suffices to set $\tau_S(\mathsf{A}\phi)$ equal to $\mathsf{act}(\tau_S(\phi))$.[5]

The modal systems introduced in Wehmeier (2004, 2005) are in fact modal predicate logics rather than propositional logics. The extension of our translation results to the quantificational case is straightforward, except that, when working with variable-domain models and actualist quantification, it turns out that the indicative–subjunctive distinction must be drawn also at the level of the first-order quantifiers in order to achieve full expressive power.

To be more precise: Let $L$ be an ordinary first-order language given by a set $\mathcal{P}$ of predicate symbols, written $P, Q, R, \ldots$ (each predicate symbol $P$ being associated with a natural number $\#P$ as its arity) and a set $C$ of constant symbols.[6]

[5]For more discussion of subjunctive closure and issues arising from endowing $\mathcal{L}^1_{\mathsf{A}}$ with real-world truth, see §4.

[6]For simplicity let us assume that first-order languages have only $\neg$ and $\wedge$ as primitive connectives, and only $\exists$ as a primitive quantifier, the other connectives and quantifiers being defined in the standard ways.

The language $L^1_s$ of modal predicate logic with one subjunctive marker based on $L$ has as primitive symbols all those of $L$, plus, first, for each $P \in \mathcal{P}$, a subjunctive version $P^s$ of the same arity, obtained by superscripting the original ("indicative") $P$ with the subjunctive marker $s$; second, a subjunctive version $\exists^s$ of the first-order existential quantifier; and third, indicative and subjunctive versions of the box, i.e. $\Box$ and $\Box^s$, respectively. The atomic formulas of $L^1_s$ are constructed as usual from both indicative and subjunctive predicate symbols. Compound formulas are obtained by means of the propositional connectives, the indicative and subjunctive existential quantifiers, and the indicative and subjunctive boxes as usual.

The $L^1_s$-formulas are interpreted in models

$$\mathbf{M} = \left(W, @, R, (D_w)_{w \in W}, (P_w)^{w \in W}_{P \in \mathcal{P}}, (c^{\mathbf{M}})_{c \in C}\right),$$

where $W$, @ and $R$ are as in propositional modal logic, $(D_w)_{w \in W}$ is a family of sets indexed by $W$ such that $D := \bigcup\{D_w \mid w \in W\}$ is non-empty, $(P_w)^{w \in W}_{P \in \mathcal{P}}$ is a family doubly indexed by $W$ and $\mathcal{P}$ such that for each $w \in W$ and $P \in \mathcal{P}$ with $\#P = n$, $P_w \subseteq D^n$, and $(c^{\mathbf{M}})_{c \in C}$ is a family, indexed by $C$, of elements of $D_@$. Intuitively, $D_w$ is the set of individuals existing at $w$, $P_w$ is the extension of $P$ at the world $w$, and $c^{\mathbf{M}}$ is the object (rigidly) designated by the individual constant $c$.

In defining the notion of truth (or rather, satisfaction) at a world in such a model for $L^1_s$-formulas, we need to introduce an additional parameter, namely an assignment of values to the individual variables. Such an assignment $\alpha$ is simply a mapping from the set of individual variables into the (non-empty!) union $D$ of the worldly domains $D_w$. Where $\alpha$ is a variable assignment, $x$ a variable, and $a \in D$, we let $\alpha^a_x$ be the assignment that agrees with $\alpha$ on all arguments other than $x$ and maps $x$ to $a$. For ease of notation, we also introduce the convention that $\mathbf{M}_\alpha(t)$ equals $\alpha(t)$ when $t$ is an individual variable, and equals $c^{\mathbf{M}}$ when $t$ is the individual constant $c$. The definition of satisfaction at a world $w$ by an assignment $\alpha$ now proceeds by recursion on $L^1_s$-formulas, as follows:

1. $\mathbf{M}, \alpha \models_w Pt_1 \ldots t_n$ if and only if $\langle \mathbf{M}_\alpha(t_1), \ldots, \mathbf{M}_\alpha(t_n) \rangle \in P_@$

2. $\mathbf{M}, \alpha \models_w P^s t_1 \ldots t_n$ if and only if $\langle \mathbf{M}_\alpha(t_1), \ldots, \mathbf{M}_\alpha(t_n) \rangle \in P_w$

3. $\mathbf{M}, \alpha \models_w \neg\phi$ if and only if $\mathbf{M}, \alpha \not\models_w \phi$

4. $\mathbf{M}, \alpha \models_w (\phi \wedge \psi)$ if and only if $\mathbf{M}, \alpha \models_w \phi$ and $\mathbf{M}, \alpha \models_w \psi$

5. $\mathbf{M}, \alpha \models_w \Box\phi$ if and only if for all $v$ with $@Rv$, $\mathbf{M}, \alpha \models_v \phi$

6. $\mathbf{M}, \alpha \models_w \Box^s \phi$ if and only if for all $v$ with $wRv$, $\mathbf{M}, \alpha \models_v \phi$

7. $\mathbf{M}, \alpha \models_w \exists x \phi$ if and only if for some $a \in D_@$, $\mathbf{M}, \alpha^a_x \models_w \phi$

8. $\mathbf{M}, \alpha \models_w \exists^s x \phi$ if and only if for some $a \in D_w$, $\mathbf{M}, \alpha^a_x \models_w \phi$

In Tarskian fashion, we will say that an individually closed formula $\phi$ of $L^1_s$, that is, a formula containing no free occurrences of variables, is true at $w$ in $\mathbf{M}$ (in symbols: $\mathbf{M} \models_w \phi$), just in case it is satisfied at $w$ by every (equivalently, any) variable assignment $\alpha$.

On the actuality side, we can define a counterpart $L^1_{\mathsf{A}}$ to $L^1_s$ as follows. Its primitive symbols are those of $L$ together with the box $\Box$, the actuality operator $\mathsf{A}$, and the existential actuality quantifier $\exists^{\mathsf{A}}$. Formulas are constructed exactly as one would expect, with the standard and actuality quantifier deployed in exactly the same way. The $L^1_{\mathsf{A}}$-formulas are evaluated at worlds $w$ and relative to variable assignments $\alpha$ in models $\mathbf{M} = (W, @, R, (D_w)_{w \in W}, (P_w)^{w \in W}_{P \in \mathcal{P}}, (c^{\mathbf{M}})_{c \in C})$, just like the $L^1_s$-formulas. The recursive clauses for satisfaction of $\phi$ by $\alpha$ at $w$ in $\mathbf{M}$ (in symbols: $\mathbf{M}, \alpha \Vdash_w \phi$) run as follows.

1. $\mathbf{M}, \alpha \Vdash_w Pt_1 \ldots t_n$ if and only if $\langle \mathbf{M}_\alpha(t_1), \ldots, \mathbf{M}_\alpha(t_n) \rangle \in P_w$

2. $\mathbf{M}, \alpha \Vdash_w \neg \phi$ if and only if $\mathbf{M}, \alpha \nVdash_w \phi$

3. $\mathbf{M}, \alpha \Vdash_w (\phi \wedge \psi)$ if and only if $\mathbf{M}, \alpha \Vdash_w \phi$ and $\mathbf{M}, \alpha \Vdash_w \psi$

4. $\mathbf{M}, \alpha \Vdash_w \Box \phi$ if and only if for all $v$ with $wRv$, $\mathbf{M}, \alpha \Vdash_v \phi$

5. $\mathbf{M}, \alpha \Vdash_w \exists x \phi$ if and only if for some $a \in D_w$, $\mathbf{M}, \alpha^a_x \Vdash_w \phi$

6. $\mathbf{M}, \alpha \Vdash_w \exists^{\mathsf{A}} x \phi$ if and only if for some $a \in D_@$, $\mathbf{M}, \alpha^a_x \Vdash_w \phi$

7. $\mathbf{M}, \alpha \Vdash_w \mathsf{A} \phi$ if and only if $\mathbf{M}, \alpha \Vdash_@ \phi$

As in the case of $L^1_s$, we define *truth* at $w$ in $\mathbf{M}$ for individually closed $L^1_{\mathsf{A}}$-formulas only, where a formula is individually closed if it contains no free occurrences of variables, just as in the case of $L^1_s$. For such an individually closed $L^1_{\mathsf{A}}$-formula $\phi$, we say that $\phi$ is true at $w$ in $\mathbf{M}$, in symbols $\mathbf{M} \Vdash_w \phi$, if $\phi$ is satisfied at $w$ by every (equivalently, any) variable assignment $\alpha$.

The translation $t_{s,\mathsf{A}}$ from $\mathcal{L}^1_s$ to $\mathcal{L}^1_{\mathsf{A}}$ can be extended in a straightforward way to a translation from $L^1_s$ to $L^1_{\mathsf{A}}$, as follows.

- $t_{s,\mathsf{A}}(Pt_1 \ldots t_n) = \mathsf{A}Pt_1 \ldots t_n$

- $t_{s,\mathsf{A}}(P^s t_1 \dots t_n) = Pt_1 \dots t_n$
- $t_{s,\mathsf{A}}(\neg\phi) = \neg t_{s,\mathsf{A}}(\phi)$
- $t_{s,\mathsf{A}}(\phi \wedge \psi) = (t_{s,\mathsf{A}}(\phi) \wedge t_{s,\mathsf{A}}(\psi))$
- $t_{s,\mathsf{A}}(\Box\phi) = \mathsf{A}\Box t_{s,\mathsf{A}}(\phi)$
- $t_{s,\mathsf{A}}(\Box^s\phi) = \Box t_{s,\mathsf{A}}(\phi)$
- $t_{s,\mathsf{A}}(\exists x\phi) = \exists^{\mathsf{A}} x\, t_{s,\mathsf{A}}(\phi)$
- $t_{s,\mathsf{A}}(\exists^s x\phi) = \exists x\, t_{s,\mathsf{A}}(\phi)$

We leave it to the reader to extend lemma 1 to the quantified case, i.e. to show that, for all $L^1_s$-formulas $\phi$, all models $\mathbf{M} = (W, @, R, (D_w)_{w\in W}, (P_w)^{w\in W}_{P\in\mathcal{P}}, (c^{\mathbf{M}})_{c\in C})$, all worlds $w \in W$, and all variable assignments $\alpha$ in $\mathbf{M}$:

$$\mathbf{M}, \alpha \models_w \phi \text{ if and only if } \mathbf{M}, \alpha \Vdash_w t_{s,\mathsf{A}}(\phi).$$

Translation in the direction from $L^1_{\mathsf{A}}$ to $L^1_s$, too, can be effected along lines parallel to the propositional case. To this end, we first extend the $\mathsf{A}$-simulating mapping $\mathsf{act}$ to $L^1_s$ as follows.

- $\mathsf{act}(Pt_1 \dots t_n) = \mathsf{act}(P^s t_1 \dots t_n) = Pt_1 \dots t_n$
- $\mathsf{act}(\neg\phi) = \neg\mathsf{act}(\phi)$
- $\mathsf{act}(\phi \wedge \psi) = (\mathsf{act}(\phi) \wedge \mathsf{act}(\psi))$
- $\mathsf{act}(\Box\phi) = \mathsf{act}(\Box^s\phi) = \Box\phi$
- $\mathsf{act}(\exists x\phi) = \mathsf{act}(\exists^s x\phi) = \exists x\, \mathsf{act}(\phi)$

The reader will have no trouble to establish the following analogue to lemma 2: For every $L^1_s$-formula $\phi$, every model $\mathbf{M}$, every world $w$ of $\mathbf{M}$, and every variable assignment $\alpha$ in $\mathbf{M}$,

$$\mathbf{M}, \alpha \models_w \mathsf{act}(\phi) \text{ if and only if } \mathbf{M}, \alpha \models_@ \phi.$$

With the help of $\mathsf{act}$ we may then, as in the propositional case, define the translation $t_{\mathsf{A},s}$ from $L^1_{\mathsf{A}}$ to $L^1_s$, as follows.

- $t_{\mathsf{A},s}(Pt_1 \dots t_n) = P^s t_1 \dots t_n$
- $t_{\mathsf{A},s}(\neg\phi) = \neg t_{\mathsf{A},s}(\phi)$
- $t_{\mathsf{A},s}(\phi \wedge \psi) = (t_{\mathsf{A},s}(\phi) \wedge t_{\mathsf{A},s}(\psi))$
- $t_{\mathsf{A},s}(\Box\phi) = \Box^s t_{\mathsf{A},s}(\phi)$
- $t_{\mathsf{A},s}(\mathsf{A}\phi) = \mathsf{act}(t_{\mathsf{A},s}(\phi))$
- $t_{\mathsf{A},s}(\exists x\phi) = \exists^s x\, t_{\mathsf{A},s}(\phi)$
- $t_{\mathsf{A},s}(\exists^{\mathsf{A}} x\phi) = \exists x\, t_{\mathsf{A},s}(\phi)$

As in lemma 3, one can then show that for every $L^1_{\mathsf{A}}$-formula $\phi$, every model $\mathbf{M}$, every world $w$ of $\mathbf{M}$, and every variable assignment $\alpha$ in $\mathbf{M}$,

$$\mathbf{M}, \alpha \Vdash_w \phi \text{ if and only if } \mathbf{M}, \alpha \models_w t_{\mathsf{A},s}(\phi).$$

We note that both $t_{s,\mathsf{A}}$ and $t_{\mathsf{A},s}$ map individually closed formulas of the one language to individually closed formulas of the other, and so we also have that for every individually closed $L^1_{\mathsf{A}}$-formula $\phi$, every individually closed $L^1_s$-formula $\psi$, every model $\mathbf{M}$, and every world $w$ of $\mathbf{M}$,

$$\mathbf{M} \Vdash_w \phi \text{ if and only if } \mathbf{M} \models_w t_{\mathsf{A},s}(\phi)$$

and

$$\mathbf{M} \Vdash_w t_{s,\mathsf{A}}(\psi) \text{ if and only if } \mathbf{M} \models_w \psi.$$

Call an $L^1_s$-formula $\phi$ subjunctively closed if every occurrence of a subjunctivized predicate symbol, of a subjunctive quantifier, and of a subjunctive box lies within the scope of a box in $\phi$. As in the propositional case, we have that the evaluation of a subjunctively closed formula is insensitive to the world of evaluation, i.e. we have that for every subjunctively closed $L^1_{\mathsf{A}}$-formula $\phi$, every model $\mathbf{M}$, any worlds $w$ and $v$ of $\mathbf{M}$, and every variable assignment $\alpha$ in $\mathbf{M}$,

$$\mathbf{M}, \alpha \models_w \phi \text{ if and only if } \mathbf{M}, \alpha \models_v \phi.$$

For subjunctively closed $L^1_s$-formulas $\phi$ we can therefore define a notion of satisfaction by a variable assignment $\alpha$ in a model $\mathbf{M}$ as follows: $\phi$ is satisfied by $\alpha$ in $\mathbf{M}$, $\mathbf{M}, \alpha \models \phi$, if $\alpha$ satisfies $\phi$ at every world $w$ of $\mathbf{M}$ (equivalently, at some world or other of $\mathbf{M}$, equivalently, at the actual world of $\mathbf{M}$).

If an $L^1_s$-formula $\phi$ is both subjunctively and individually closed—in which case we will call $\phi$ an $L^1_s$-*sentence*—its evaluation in a model is independent both of the world of evaluation and the variable assignment, so we can define a notion of truth *simpliciter* for it: Let's say that an $L^1_s$-sentence is true in a model $\mathbf{M}$ (for short, $\mathbf{M} \models \phi$), just in case it is satisfied by every assignment (equivalently: some assignment or other) at every world of $\mathbf{M}$ (equivalently, at some world or other of $\mathbf{M}$, equivalently, at the actual world of $\mathbf{M}$).

Let's say further that an $L^1_{\mathsf{A}}$-formula $\phi$ is satisfied by a variable assignment $\alpha$ in a model $\mathbf{M}$ (for short, $\mathbf{M}, \alpha \Vdash \phi$) just in case $\phi$ is satisfied by $\alpha$ at the actual world of $\mathbf{M}$ (i.e. $\mathbf{M}, \alpha \Vdash_{@} \phi$), and that an individually closed $L^1_{\mathsf{A}}$-formula $\phi$ is true in $\mathbf{M}$ (for short, $\mathbf{M} \Vdash \phi$) just in case $\phi$ is satisfied by every assignment (equivalently, some assignment or other) at the actual world of $\mathbf{M}$. Noting that the range of $\mathsf{act}$ consists entirely of subjunctively closed $L^1_s$-formulas, we then have the following results:

- For every $L^1_{\mathsf{A}}$-formula $\phi$, every model $\mathbf{M}$, and every assignment $\alpha$,

$$\mathbf{M}, \alpha \Vdash \phi \text{ if and only if } \mathbf{M}, \alpha \models \mathsf{act} \circ t_{\mathsf{A},s}(\phi).$$

- For every individually closed $L^1_{\mathsf{A}}$-formula $\phi$ and every model $\mathbf{M}$,

$$\mathbf{M} \Vdash \phi \text{ if and only if } \mathbf{M} \models \mathsf{act} \circ t_{\mathsf{A},s}(\phi).$$

# 3 Infinitely Many Subjunctive Markers

As has been observed in the literature (e.g. Peacocke 1978; Forbes 1985, 1989), (the quantified version of) $\mathcal{L}^1_{\mathsf{A}}$ still suffers from certain expressive deficits which can be overcome by supplementing the language with indexed boxes and actuality operators. Here we show—for simplicity, in the propositional setting only—that an analogous extension of $\mathcal{L}^1_s$ is available.

We begin by defining the language $\mathcal{L}^\infty_s$ of propositional modal logic with infinitely many subjunctive markers; afterwards we formulate the corresponding actuality-based system $\mathcal{L}^\infty_{\mathsf{A}}$ and provide translation results.

**Definition 5** The primitive symbols of $\mathcal{L}^\infty_s$ are, besides parentheses for grouping,

1. the set $\mathsf{Sent}$ of sentence letters;
2. for each natural number $i \geq 0$ and each $p \in \mathsf{Sent}$, the $i$-subjunctivized sentence letter $p^i$;

3. the connectives $\neg$ and $\wedge$;

4. for each $i, j \geq 0$, the indicative box $\Box_j$ indexed by $j$ and the $i$-subjunctivized box $\Box^i_j$ indexed by $j$.

The formulas of $\mathcal{L}^\infty_s$ are generated inductively from indicative and subjunctivized sentence letters as atoms by means of the propositional connectives and the indicative and subjunctivized indexed boxes as usual.

The language $\mathcal{L}^\infty_s$ is interpreted in the familiar possible-world models. The central semantic notion here is that of an $\mathcal{L}^\infty_s$-formula's being *true relative to* an infinite sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$ of worlds, written $\mathbf{M} \models^\sigma \phi$.[7] It is defined recursively as follows:

- $\mathbf{M} \models^\sigma p$ iff $V_@(p) = 1$
- $\mathbf{M} \models^\sigma p^i$ iff $V_{\sigma_i}(p) = 1$
- $\mathbf{M} \models^\sigma \neg\phi$ iff $\mathbf{M} \not\models^\sigma \phi$
- $\mathbf{M} \models^\sigma (\phi \wedge \psi)$ iff $\mathbf{M} \models^\sigma \phi$ and $\mathbf{M} \models^\sigma \psi$
- $\mathbf{M} \models^\sigma \Box_i\phi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{\sigma^v_i} \phi$
- $\mathbf{M} \models^\sigma \Box^j_i\phi$ iff for all $v$ with $\sigma_j Rv$, $\mathbf{M} \models^{\sigma^v_i} \phi$

For purposes of comparison, we next introduce the language $\mathcal{L}^\infty_{\mathsf{A}}$ of propositional modal logic with infinitely many actuality operators. It has as primitive symbols all those of $\mathcal{L}^1_{\mathsf{A}}$, and in addition an indexed box $\Box_i$ and an indexed actuality operator $\mathsf{A}_i$ for each natural number $i \geq 0$. The new operators behave syntactically just like their unindexed versions. Like $\mathcal{L}^\infty_s$, $\mathcal{L}^\infty_{\mathsf{A}}$ is interpreted in models $\mathbf{M} = (W, @, R, \mathcal{V})$ relative to an infinite sequence $\sigma$ of worlds from $W$. In contradistinction to the case of $\mathcal{L}^\infty_s$, we use an additional single-world parameter for the evaluation of $\mathcal{L}^\infty_{\mathsf{A}}$-formulas that will be called, for obvious reasons, the *world of evaluation*.

The notion of an $\mathcal{L}^\infty_{\mathsf{A}}$-formula $\phi$ being true at a world $w$ relative to a world sequence $\sigma$ in the model $\mathbf{M} = (W, @, R, \mathcal{V})$, $\mathbf{M} \Vdash^\sigma_w \phi$, is defined recursively as follows:

[7]Where $w \in W$, we let $\sigma^w_i$ be the sequence that is exactly like $\sigma$ in all places other than the $i$th, where $\sigma^w_i$ has $w$.

1. $\mathbf{M} \Vdash^{\sigma}_{w} p$ iff $V_w(p) = 1$

2. $\mathbf{M} \Vdash^{\sigma}_{w} \neg\phi$ iff $\mathbf{M} \nVdash^{\sigma}_{w} \phi$

3. $\mathbf{M} \Vdash^{\sigma}_{w} (\phi \wedge \psi)$ iff $\mathbf{M} \Vdash^{\sigma}_{w} \phi$ and $\mathbf{M} \Vdash^{\sigma}_{w} \psi$

4. $\mathbf{M} \Vdash^{\sigma}_{w} \mathsf{A}\phi$ iff $\mathbf{M} \Vdash^{\sigma}_{@} \phi$

5. $\mathbf{M} \Vdash^{\sigma}_{w} \Box\phi$ iff for all $v \in W$ with $wRv$, $\mathbf{M} \Vdash^{\sigma}_{v} \phi$

6. $\mathbf{M} \Vdash^{\sigma}_{w} \mathsf{A}_i\phi$ iff $\mathbf{M} \Vdash^{\sigma}_{\sigma_i} \phi$

7. $\mathbf{M} \Vdash^{\sigma}_{w} \Box_i\phi$ iff for all $v \in W$ with $wRv$, $\mathbf{M} \Vdash^{\sigma^v_i}_{v} \phi$

This is essentially the system proposed by Forbes (1985, pp. 91-2, footnote 28; 1989, p. 88) in elaboration of some suggestions by Peacocke (1978). Our presentation here is slightly more general in that Forbes invariably works with a universal accessibility relation, mention of which he therefore suppresses.

An immediate difficulty in comparing $\mathcal{L}^{\infty}_{\mathsf{A}}$ and $\mathcal{L}^{\infty}_{s}$ is that the former evaluates formulas with respect to two parameters, a world of evaluation and a world sequence, whereas the latter uses only a world sequence. This is due to a combination of two features of $\mathcal{L}^{\infty}_{\mathsf{A}}$ that require singling out a particular world of evaluation. First, since $\mathcal{L}^{\infty}_{\mathsf{A}}$ does not treat a naked sentence letter $p$ as making reference to the actual world, it is simply undetermined what a notation like $\mathbf{M} \Vdash^{\sigma} p$ could mean, since all entries in the sequence $\sigma$ are on a par.[8] Similarly, since the operators $\mathsf{A}_i$ may have compound formulas in their scopes, the particular worlds in the sequence $\sigma$ to which they point must be placed in a special storage (i.e. the space to the lower right of the turnstile), otherwise it would be impossible to distinguish $\mathsf{A}_i\phi$ from $\mathsf{A}_j\phi$.[9]

Since formulas are finite sequences of symbols, there is, for each $\mathcal{L}^{\infty}_{\mathsf{A}}$-formula $\phi$, a smallest number $n(\phi)$ such that each index and superscript occurring in $\phi$ is less than $n(\phi)$. When translating $\phi$ into $\mathcal{L}^{\infty}_{s}$, we can then use the $n(\phi)$-th place of the sequence $\sigma$, which is addressed by the subjunctive marker $n(\phi)$ in the $\mathcal{L}^{\infty}_{s}$

[8] It won't do to assign the first entry in $\sigma$ a special role, since that just means designating it as specifying the world of evaluation.

[9] If only sentence letters could ever occur within the scopes of the $\mathsf{A}_i$, this problem wouldn't arise, as we could simply stipulate that $\mathbf{M} \Vdash^{\sigma} \mathsf{A}_i p$ be equivalent to $V_{\sigma_i}(p) = 1$, which is essentially what happens in $\mathcal{L}^{\infty}_{s}$. But already for $\mathbf{M} \Vdash^{\sigma} \mathsf{A}_i(p \wedge q)$ we must remember which world the index $i$ points to.

framework, to simulate the world of evaluation.[10] Preparatory to the definition of a translation from $\mathcal{L}_{\mathsf{A}}^{\infty}$ to $\mathcal{L}_{s}^{\infty}$, we therefore now define the free occurrences of the subjunctive marker $k$ (for short: free occurrences of $k$) in an $\mathcal{L}_{s}^{\infty}$-formula $\phi$ by recursion on $\phi$.

- There are no free occurrences of $k$ in any indicative sentence letter $p$, or in any sentence letter $p^i$ superscripted with a natural number $i \neq k$.
- The one occurrence of $k$ in $p^k$ is a free occurrence of $k$ in $p^k$.
- The free occurrences of $k$ in $\neg\phi$ are the free occurrences of $k$ in $\phi$.
- The free occurrences of $k$ in $(\phi \wedge \psi)$ are the free occurrences of $k$ in $\phi$ and the free occurrences of $k$ in $\psi$.
- There are no free occurrences of $k$ in $\Box_k\phi$ or in $\Box_k^j\phi$ as long as $j \neq k$.
- There is exactly one free occurrence of $k$ in $\Box_k^k\phi$, and that is the one to the upper right of the initial box.
- Where $i, j \neq k$, the free occurrences of $k$ in $\Box_i\phi$ and in $\Box_i^j\phi$ are those in $\phi$.
- Where $i \neq k$, the free occurrences of $k$ in $\Box_i^k\phi$ are those in $\phi$ together with the occurrence of $k$ to the upper right of the initial box.

We need to fix two more auxiliary notions before we can embark on translation: the binary operation $(\phi, i) \mapsto \mathsf{Er}(\phi, i)$, defined on pairs of $\mathcal{L}_{s}^{\infty}$-formulas $\phi$ and subjunctive markers $i$, that erases all free occurrences of the marker $i$ in $\phi$, and the ternary substitution operation $(\phi, j, k) \mapsto \phi_j[k]$, defined on triples $(\phi, j, k)$ of $\mathcal{L}_{s}^{\infty}$-formulas $\phi$, subjunctive markers $j$, and subjunctive markers $k$, that replaces all free occurrences of the marker $j$ in $\phi$ by $k$.

For each $n \geq 0$, we first define the pre-translation with respect to $n$, $T_{\mathsf{A},s}^{n}$, as a function from $\mathcal{L}_{\mathsf{A}}^{\infty}$ to $\mathcal{L}_{s}^{\infty}$, by recursion on the $\mathcal{L}_{\mathsf{A}}^{\infty}$-formulas.

[10] As an anonymous reviewer for this journal points out, the use of the $n(\phi)$-th position in $\sigma$ for simulating the world of evaluation is somewhat analogous to Stephanou's (2001) use of $n(\phi)$ to simulate evaluation at the actual world. Unfortunately we don't have space to discuss the issue at any length, but it should be noted that Stephanou uses this device to dispense with the unindexed actuality operator entirely, assuming instead that truth in a model is truth relative to the sequence $\sigma$ that has the actual world occurring in every position, so that $n(\phi)$ points to the actual world through $\sigma$. However, we note that on this way of eliminating the unadorned operator $\mathsf{A}$, one no longer has any formulas in the modal language whose truth value in a model is always independent of the world sequence $\sigma$; in other words, eliminating the naked actuality operator also eliminates the possibility of subjunctive closure. This strikes us as an undesirable consequence.

- $T^n_{\mathsf{A},s}(p) = p^n$
- $T^n_{\mathsf{A},s}(\neg\phi) = \neg T^n_{\mathsf{A},s}(\phi)$
- $T^n_{\mathsf{A},s}(\phi \wedge \psi) = (T^n_{\mathsf{A},s}(\phi) \wedge T^n_{\mathsf{A},s}(\psi))$
- $T^n_{\mathsf{A},s}(\mathsf{A}\phi)$ is the result $\mathsf{Er}(T^n_{\mathsf{A},s}(\phi), n)$ of erasing all free occurrences of $n$ in $T^n_{\mathsf{A},s}(\phi)$
- $T^n_{\mathsf{A},s}(\Box\phi) = \Box^n_n T^n_{\mathsf{A},s}(\phi)$
- $T^n_{\mathsf{A},s}(\mathsf{A}_i\phi)$ is the result $\left(T^n_{\mathsf{A},s}(\phi)\right)_n [i]$ of replacing all free occurrences of $n$ in $T^n_{\mathsf{A},s}(\phi)$ with occurrences of $i$
- $T^n_{\mathsf{A},s}(\Box_i\phi)$ is the result $\Box^n_i \left(T^n_{\mathsf{A},s}(\phi)\right)_n [i]$ of prefixing $\Box^n_i$ to the result $\left(T^n_{\mathsf{A},s}(\phi)\right)_n [i]$ of replacing all free occurrences of $n$ in $T^n_{\mathsf{A},s}(\phi)$ by $i$

Now define, for $\mathcal{L}^\infty_{\mathsf{A}}$-formulas $\phi$, the translation $T_{\mathsf{A},s}(\phi)$ of $\phi$ into $\mathcal{L}^\infty_s$ as $T^{n(\phi)}_{\mathsf{A},s}(\phi)$, where $n(\phi)$ is, as above, the least number greater than every index and superscript occurring in $\phi$.

Our next aim is to prove the following result.

**Proposition 1** *For all $\mathcal{L}^\infty_{\mathsf{A}}$-formulas $\phi$, all models $\mathbf{M} = (W, @, R, \mathcal{V})$, all sequences $\sigma \in W^{\mathbb{N}}$ and all $w \in W$:*

$$\mathbf{M} \Vdash^\sigma_w \phi \quad \textit{iff} \quad \mathbf{M} \models^{\sigma^w_{n(\phi)}} T_{\mathsf{A},s}(\phi).$$

To this end, we need a few preparatory lemmas.

**Lemma 4** *Let $\mathbf{M} = (W, @, R, \mathcal{V})$ be a model, let $\phi$ be an $\mathcal{L}^\infty_s$-formula, and let $\sigma, \tau \in W^{\mathbb{N}}$ be such that $\sigma_i = \tau_i$ for every $i$ that has a free occurrence in $\phi$. Then $\mathbf{M} \models^\sigma \phi$ if and only if $\mathbf{M} \models^\tau \phi$.*

The proof proceeds by induction on $\phi$ and is familiar from the result in first-order logic according to which the satisfaction of a formula by an assignment depends only on the values assigned to variables occurring free in the formula at hand.

**Lemma 5** *For every $\mathcal{L}^\infty_s$-formula $\phi$, model $\mathbf{M} = (W, @, R, \mathcal{V})$, $\sigma \in W^{\mathbb{N}}$, $n \in \mathbb{N}$, and $w \in W$:*

$$\mathbf{M} \models^{\sigma^@_n} \phi \quad \textit{iff} \quad \mathbf{M} \models^{\sigma^w_n} \mathsf{Er}(\phi, n).$$

The proof proceeds by induction on $\phi$.

For naked sentence letters $p$, we have that $\mathbf{M} \models^{\sigma_n^@} p$ iff $V_@(p) = 1$ iff $\mathbf{M} \models^{\sigma_n^w} p$ (the world sequence does not enter into the evaluation of a naked sentence letter). But of course $\mathsf{Er}(p,n)$ is simply $p$, hence this final condition is equivalent to $\mathbf{M} \models^{\sigma_n^w} \mathsf{Er}(p,n)$.

For subjunctivized sentence letters $p^i$, $i \neq n$, we have that $\mathbf{M} \models^{\sigma_n^@} p^i$ iff $V_{\sigma_i}(p) = 1$, because the $i$-th entry in $\sigma_n^@$ is $\sigma_i$. But this is equivalent to $\mathbf{M} \models^{\sigma_n^w} p^i$, because the $i$-th entry in $\sigma_n^w$ is also $\sigma_i$. This is what we had to show, because for $i \neq n$, $\mathsf{Er}(p^i,n)$ is just $p^i$.

For subjunctivized sentence letters $p^n$, we have that $\mathbf{M} \models^{\sigma_n^@} p^n$ iff $V_@(p) = 1$, because the $n$-th entry in $\sigma_n^@$ is $@$. This is equivalent to $\mathbf{M} \models^{\sigma_n^w} p$, and that's what we needed to show, since $\mathsf{Er}(p^n,n)$ is $p$.

The cases of negations and conjunctions are easily handled by means of the inductive hypothesis.

For formulas of the form $\Box_i\psi$ with $i \neq n$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_i\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^@)_i^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_i^v)_n^@} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_i^v)_n^w} \mathsf{Er}(\psi,n)$ (induction hypothesis) iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^w)_i^v} \mathsf{Er}(\psi,n)$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_i\mathsf{Er}(\psi,n)$, and $\Box_i\mathsf{Er}(\psi,n)$ is $\mathsf{Er}(\Box_i\psi,n)$.

For formulas of the form $\Box_n\psi$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_n\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^@)_n^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^w)_n^v} \psi$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_n\psi$, and $\Box_n\psi$ is $\mathsf{Er}(\Box_n\psi,n)$.

For formulas of the form $\Box_i^j\psi$ with $i, j \neq n$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_i^j\psi$ iff for all $v$ with $(\sigma_n^@)_jRv$, $\mathbf{M} \models^{(\sigma_n^@)_i^v} \psi$ iff for all $v$ with $(\sigma_n^w)_jRv$, $\mathbf{M} \models^{(\sigma_i^v)_n^@} \psi$ iff for all $v$ with $(\sigma_n^w)_jRv$, $\mathbf{M} \models^{(\sigma_i^v)_n^w} \mathsf{Er}(\psi,n)$ (induction hypothesis) iff for all $v$ with $(\sigma_n^w)_jRv$, $\mathbf{M} \models^{(\sigma_n^w)_i^v} \mathsf{Er}(\psi,n)$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_i^j\mathsf{Er}(\psi,n)$, and $\Box_i^j\mathsf{Er}(\psi,n)$ is $\mathsf{Er}(\Box_i^j\psi,n)$.

For formulas of the form $\Box_i^n\psi$ with $i \neq n$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_i^n\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^@)_i^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_i^v)_n^@} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_i^v)_n^w} \mathsf{Er}(\psi,n)$ (induction hypothesis) iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^w)_i^v} \mathsf{Er}(\psi,n)$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_i\mathsf{Er}(\psi,n)$, and $\Box_i\mathsf{Er}(\psi,n)$ is $\mathsf{Er}(\Box_i^n\psi,n)$.

For formulas of the form $\Box_n^j\psi$ with $j \neq n$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_n^j\psi$ iff for all $v$ with $(\sigma_n^@)_jRv$, $\mathbf{M} \models^{(\sigma_n^@)_n^v} \psi$ iff for all $v$ with $(\sigma_n^w)_jRv$, $\mathbf{M} \models^{(\sigma_n^w)_n^v} \psi$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_n^j\psi$, and $\Box_n^j\psi$ is $\mathsf{Er}(\Box_n^j\psi,n)$.

Finally, for formulas of the form $\Box_n^n\psi$ we have that $\mathbf{M} \models^{\sigma_n^@} \Box_n^n\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^@)_n^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^w)_n^v} \psi$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_n\psi$, and $\Box_n\psi$ is $\mathsf{Er}(\Box_n^n\psi,n)$, QED.

For the next lemma, we need to define the notion of substitutibility, familiar from first-order logic, in the context of $\mathcal{L}_s^\infty$.

**Definition 6** Let $\phi$ be an $\mathcal{L}_s^\infty$-formula and $i, n \in \mathbb{N}$. We define the condition of $i$ being substitutible for $n$ in $\phi$ by recursion on $\phi$, as follows.

1. If $\phi$ is atomic (i.e. a naked or a subjunctivized sentence letter), $i$ is substitutible for $n$ in $\phi$.

2. If $\phi$ is a negation $\neg\psi$, $i$ is substitutible for $n$ in $\phi$ iff $i$ is substitutible for $n$ in $\psi$.

3. If $\phi$ is a conjunction $(\psi \wedge \theta)$, $i$ is substitutible for $n$ in $\phi$ iff $i$ is substitutible for $n$ in $\psi$ and in $\theta$.

4. If $\phi$ is of the form $\Box_n\psi$ or $\Box_n^j\psi$, $i$ is substitutible for $n$ in $\phi$.

5. If $\phi$ is of the form $\Box_k\psi$ or $\Box_k^j\psi$ with $k \neq n$, then $i$ is substitutible for $n$ in $\phi$ iff $k \neq i$ and $i$ is substitutible for $n$ in $\psi$.

**Lemma 6** *For every $\mathcal{L}_s^\infty$-formula $\phi$, model $\mathbf{M} = (W, @, R, \mathcal{V})$, $\sigma \in W^{\mathbb{N}}$, $w \in W$, and $n, i \in \mathbb{N}$ such that i is substitutible for n in $\phi$:*

$$\mathbf{M} \models^{\sigma_n^{\sigma_i}} \phi \quad \textit{iff} \quad \mathbf{M} \models^{\sigma} \phi_n[i].$$

The proof proceeds by induction on $\phi$.

If $\phi$ is a naked sentence letter $p$, the formula $\phi_n[i]$ is just $p$, and both sides of the equivalence reduce to $V_@(p) = 1$.

If $\phi$ is a subjunctivized sentence letter $p^j$ with $j \neq n$, the formula $\phi_n[i]$ is just $p^j$, and both sides of the equivalence reduce to $V_{\sigma_j}(p) = 1$.

If $\phi$ is a subjunctivized sentence letter $p^n$, the formula $\phi_n[i]$ is $p^i$, and we have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} p^n$ iff $V_{(\sigma_n^{\sigma_i})_n}(p) = 1$. But of course $(\sigma_n^{\sigma_i})_n = \sigma_i$, and so this is the case iff $V_{\sigma_i}(p) = 1$, which holds iff $\mathbf{M} \models^{\sigma} p^i$, as required.

If $\phi$ is of the form $\Box_n\psi$, the formula $\phi_n[i]$ is just $\phi$ itself, and we have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_n\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_n^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{\sigma_n^v} \psi$ iff $\mathbf{M} \models^{\sigma} \Box_n\psi$, as required.

If $\phi$ is of the form $\Box_n^k\psi$ with $k \neq n$, the formula $\phi_n[i]$ is just $\phi$ itself, and we have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_n^k\psi$ iff for all $v$ with $\sigma_k Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_n^v} \psi$ iff for all $v$ with $\sigma_k Rv$, $\mathbf{M} \models^{\sigma_n^v} \psi$ iff $\mathbf{M} \models^{\sigma} \Box_n^k\psi$, as required.

If $\phi$ is of the form $\Box_n^n\psi$, the formula $\phi_n[i]$ is $\Box_n^i\psi$, and we have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_n^n\psi$ iff for all $v$ with $\sigma_i Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_n^v} \psi$ iff for all $v$ with $\sigma_i Rv$, $\mathbf{M} \models^{\sigma_n^v} \psi$ iff $\mathbf{M} \models^{\sigma} \Box_n^i\psi$, as required.

If $\phi$ is of the form $\Box_j\psi$ with $j \neq n$, substitutibility implies that $i \neq j$, the formula $\phi_n[i]$ is $\Box_j(\psi_n[i])$, and we have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_j\psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_j^v} \psi$ iff for all $v$ with $@Rv$, $\mathbf{M} \models^{(\sigma_j^v)_n^{\sigma_i}} \psi$ (recall that $i \neq j$) iff for all $v$ with $@Rv$, $\mathbf{M} \models^{\sigma_j^v} \psi_n[i]$ (induction hypothesis) iff $\mathbf{M} \models^{\sigma} \Box_j(\psi_n[i])$, as required.

If $\phi$ is of the form $\Box_j^k\psi$ with $j, k \neq n$, substitutibility implies that $i \neq j$, and the formula $\phi_n[i]$ is $\Box_j^k(\psi_n[i])$. We have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_j^k\psi$ iff for all $v$ with $(\sigma_n^{\sigma_i})_k Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_j^v} \psi$ iff for all $v$ with $\sigma_k Rv$, $\mathbf{M} \models^{(\sigma_j^v)_n^{\sigma_i}} \psi$ (because $i \neq j$) iff for all $v$ with $\sigma_k Rv$, $\mathbf{M} \models^{(\sigma_j^v)} \psi_n[i]$ (induction hypothesis) iff $\mathbf{M} \models^{\sigma} \Box_j^k(\psi_n[i])$, as required.

If, finally, $\phi$ is of the form $\Box_j^n\psi$ with $j \neq n$, substitutibility implies that $i \neq j$, and the formula $\phi_n[i]$ is $\Box_j^i(\psi_n[i])$. We have that $\mathbf{M} \models^{\sigma_n^{\sigma_i}} \Box_j^n\psi$ iff for all $v$ with $(\sigma_n^{\sigma_i})_n Rv$, $\mathbf{M} \models^{(\sigma_n^{\sigma_i})_j^v} \psi$ iff for all $v$ with $\sigma_i Rv$, $\mathbf{M} \models^{(\sigma_j^v)_n^{\sigma_i}} \psi$ (because $i \neq j$) iff for all $v$ with $\sigma_i Rv$, $\mathbf{M} \models^{(\sigma_j^v)} \psi_n[i]$ (induction hypothesis) iff $\mathbf{M} \models^{\sigma} \Box_j^i(\psi_n[i])$, as required. QED.

The proof of proposition 1 is now easy. In fact, we prove a slightly more general result.

**Lemma 7** *For every $\mathcal{L}_{\mathsf{A}}^{\infty}$-formula $\phi$, model $\mathbf{M} = (W, @, R, \mathcal{V})$, $\sigma \in W^{\mathbb{N}}$, $w \in W$, and any $n \in \mathbb{N}$ that does not occur anywhere in $\phi$:*

$$\mathbf{M} \Vdash_w^{\sigma} \phi \quad \textit{iff} \quad \mathbf{M} \models^{\sigma_n^w} T_{\mathsf{A},s}^n(\phi).$$

Proposition 1 is an obvious corollary of lemma 7. The proof of the lemma itself proceeds by induction on $\phi$, using lemmas 4, 5, and 6.

If $\phi$ is a sentence letter $p$, $T_{\mathsf{A},s}^n\phi$ is $p^n$, and both sides of the equivalence reduce to $V_w(p) = 1$.

The cases of negations and conjunctions follow easily with the help of the induction hypothesis (but note that the conjunction case is the reason we must prove lemma 7 before proposition 1).

If $\phi$ is of the form $\Box\psi$, $T_{\mathsf{A},s}^n(\phi)$ is $\Box_n^n T_{\mathsf{A},s}^n(\psi)$, and we have that $\mathbf{M} \Vdash_w^{\sigma} \Box\psi$ iff for all $v$ with $wRv$, $\mathbf{M} \Vdash_v^{\sigma} \psi$ iff for all $v$ with $wRv$, $\mathbf{M} \models^{\sigma_n^v} T_{\mathsf{A},s}^n(\psi)$ (induction hypothesis) iff for all $v$ with $wRv$, $\mathbf{M} \models^{(\sigma_n^w)_n^v} T_{\mathsf{A},s}^n(\psi)$ iff $\mathbf{M} \models^{\sigma_n^w} \Box_n^n T_{\mathsf{A},s}^n(\psi)$, as required.

If $\phi$ is of the form $\mathsf{A}\psi$, $T_{\mathsf{A},s}^n(\phi)$ is $\mathsf{Er}(T_{\mathsf{A},s}^n(\psi), n)$, and we have that $\mathbf{M} \Vdash_w^{\sigma} \mathsf{A}\psi$ iff $\mathbf{M} \Vdash_@^{\sigma} \psi$ iff $\mathbf{M} \models^{\sigma_n^@} \psi$ (induction hypothesis) iff $\mathbf{M} \models^{\sigma_n^w} \mathsf{Er}(T_{\mathsf{A},s}^n(\psi), n)$ (by lemma 5), as required.

If $\phi$ is of the form $\Box_i\psi$, $T_{\mathsf{A},s}^n(\phi)$ is $\Box_i^n \left(T_{\mathsf{A},s}^n(\psi)\right)_n [i]$, and we have that $\mathbf{M} \Vdash_w^{\sigma} \Box_i\psi$ iff for all $v$ with $wRv$, $\mathbf{M} \Vdash_v^{\sigma_i^v} \psi$ iff for all $v$ with $wRv$, $\mathbf{M} \models^{(\sigma_i^v)_n^v} T_{\mathsf{A},s}^n(\psi)$

(induction hypothesis) iff for all $v$ with $wRv$, $\mathbf{M} \models^{\sigma_i^v} \left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$ (by lemma 6) iff for all $v$ with $wRv$, $\mathbf{M} \models^{(\sigma_n^w)_i^v} \left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$ (by lemma 4, noting that $n$ has no free occurrences in $\left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$) iff $\mathbf{M} \models^{\sigma_n^w} \Box_i^n \left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$, as required.

Finally, if $\phi$ is of the form $\mathsf{A}_i\psi$, $T^n_{\mathsf{A},s}(\phi)$ is $\left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$, and we have that $\mathbf{M} \Vdash^\sigma_w \mathsf{A}_i\psi$ iff $\mathbf{M} \Vdash^\sigma_{\sigma_i} \psi$ iff $\mathbf{M} \models^{\sigma_n^{\sigma_i}} T^n_{\mathsf{A},s}(\psi)$ (by the induction hypothesis) iff $\mathbf{M} \models^{\sigma} \left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$ (by lemma 6) iff $\mathbf{M} \models^{\sigma_n^w} \left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$ (by lemma 4, noting that $n$ has no free occurrences in $\left(T^n_{\mathsf{A},s}(\psi)\right)_n [i]$), as required. QED.

A translation $T_{s,\mathsf{A}}$ from $\mathcal{L}_s^\infty$ to $\mathcal{L}_\mathsf{A}^\infty$ can be defined very simply as follows.

- $T_{s,\mathsf{A}}(p) = \mathsf{A}p$
- $T_{s,\mathsf{A}}(p^i) = \mathsf{A}_i p$
- $T_{s,\mathsf{A}}(\neg\phi) = \neg T_{s,\mathsf{A}}(\phi)$
- $T_{s,\mathsf{A}}(\phi \wedge \psi) = (T_{s,\mathsf{A}}(\phi) \wedge T_{s,\mathsf{A}}(\psi))$
- $T_{s,\mathsf{A}}(\Box_i\phi) = \mathsf{A}\Box_i T_{s,\mathsf{A}}(\phi)$
- $T_{s,\mathsf{A}}(\Box_i^j\phi) = \mathsf{A}_j\Box_i T_{s,\mathsf{A}}(\phi)$

**Proposition 2** *For all $\mathcal{L}_s^\infty$-formulas $\phi$, all models $\mathbf{M} = (W, @, R, \mathcal{V})$, all sequences $\sigma \in W^\mathbb{N}$ and all $w \in W$:*

$$\mathbf{M} \models^\sigma \phi \quad \textit{iff} \quad \mathbf{M} \Vdash^\sigma_w T_{s,\mathsf{A}}(\phi).$$

The proof proceeds by induction on the $\mathcal{L}_s^\infty$-formula $\phi$. For naked sentence letters $p$, we have that $\mathbf{M} \models^\sigma p$ iff $V_@(p) = 1$, which is the case iff $\mathbf{M} \Vdash^\sigma_w \mathsf{A}p$ regardless of choice of $w$, and $T_{s,\mathsf{A}}(p) = \mathsf{A}p$. For subjunctive sentence letters $p^i$, all of the following are equivalent:

- $\mathbf{M} \models^\sigma p^i$
- $V_{\sigma_i}(p) = 1$
- $\mathbf{M} \Vdash^\sigma_{\sigma_i} p$
- $\mathbf{M} \Vdash^\sigma_w \mathsf{A}_i p$

and of course $T_{s,\mathsf{A}}(p^i) = \mathsf{A}_i p$. The cases of negation and conjunction follow immediately from the induction hypothesis.

For formulas of the form $\Box_i\phi$, the following are equivalent:

- $\mathbf{M} \models^{\sigma} \Box_i\phi$
- for all $v$ with $@Rv$, $\mathbf{M} \models^{\sigma_i^v} \phi$
- for all $v$ with $@Rv$, $\mathbf{M} \Vdash_v^{\sigma_i^v} T_{s,\mathsf{A}}(\phi)$ (by induction hypothesis)
- $\mathbf{M} \Vdash_{@}^{\sigma} \Box_i T_{s,\mathsf{A}}(\phi)$
- $\mathbf{M} \Vdash_{w}^{\sigma} \mathsf{A}\Box_i T_{s,\mathsf{A}}(\phi)$

and of course $T_{s,\mathsf{A}}(\Box_i\phi) = \mathsf{A}\Box_i T_{s,\mathsf{A}}(\phi)$.

Finally, for formulas of the form $\Box_i^j\phi$, the following are equivalent:

- $\mathbf{M} \models^{\sigma} \Box_i^j\phi$
- for all $v$ with $\sigma_j Rv$, $\mathbf{M} \models^{\sigma_i^v} \phi$
- for all $v$ with $\sigma_j Rv$, $\mathbf{M} \Vdash_v^{\sigma_i^v} T_{s,\mathsf{A}}(\phi)$ (by induction hypothesis)
- $\mathbf{M} \Vdash_{\sigma_j}^{\sigma} \Box_i T_{s,\mathsf{A}}(\phi)$
- $\mathbf{M} \Vdash_{w}^{\sigma} \mathsf{A}_j\Box_i T_{s,\mathsf{A}}(\phi)$

and of course $T_{s,\mathsf{A}}(\Box_i^j\phi) = \mathsf{A}_j\Box_i T_{s,\mathsf{A}}(\phi)$. QED.

# 4 Discussion

We begin by addressing a couple of points made by Humberstone (2004) that mainly concern the alleged preferability of approaches using operators, such as $\mathsf{A}$ or Humberstone's (1982) own subjunctivity operator $\mathsf{S}$, over those using mood markers.

In discussing the language $L_s^1$ as introduced in Wehmeier (2005), Humberstone (2004, 48) makes the following claim:

> It is worth noting that tense and mood behave differently in this respect, since there would, even with tensed quantifiers added, be a reduction in the expressive power of Prior-style tense logic if only tensed atomic predicates were allowed and not scope-bearing sentential tense operators. This is because of the failure, in, for the sake of simplicity, just the language with the past tense operator $\mathsf{P}$, of equivalence (1) above: we will be able, with past-tense applied only to atomic predicates, to say that at some earlier time $a$ was $F$ and at some earlier time $b$ was $F$ ($\mathsf{P}Fa \wedge \mathsf{P}Fb$) but not that at some earlier time $a$ and $b$ were both $F$ (which in Prior's language would be expressed by $\mathsf{P}(Fa \wedge Fb)$).[11]

This is quite clearly mistaken, however, as it is based on the erroneous premise that a tense logic constructed on the subjunctive marker model would contain only tensed atomic predicates but no scope-bearing sentential tense operators. Clearly, the modal language $L^1_s$ contains both atomic predicates marked for mood *and* scope-bearing modal operators, to wit, $\Box$ and $\Box^s$. An analogously constructed language for tense logic would of course also contain, besides tensed atomic predicates, scope-bearing tense operators such as Prior's $\mathsf{P}$, and $\mathsf{P}$ itself would certainly not, contrary to what Humberstone suggests, function as the tense *marker*. Thus, letting $t$ be a tense marker, one could obviously express that $a$ was $F$ at some earlier time (by $\mathsf{P}F^t a$) and $b$ was $F$ at some earlier time (by $\mathsf{P}F^t b$), but just as easily that at some earlier time, $a$ and $b$ were both $F$, to wit, by $\mathsf{P}(F^t a \wedge F^t b)$. In fact, the marker approach has the same versatility in a tense-logical setting as it does in the more strictly modal one.[12]

The passage just discussed is embedded into a larger context concerning the expressive resources of various modal languages. Humberstone there observes (2004, 47) that the mutual translatability between $\mathcal{L}^1_s$ and $\mathcal{L}^1_{\mathsf{A}}$ is due to certain strict equivalences that hold within $\mathcal{L}^1_{\mathsf{A}}$, such as $\mathsf{A}(p \wedge q) \leftrightarrow (\mathsf{A}p \wedge \mathsf{A}q)$ and $\mathsf{AA}p \leftrightarrow \mathsf{A}p$.[13] Iterating these equivalences, one can show that every $\mathcal{L}^1_{\mathsf{A}}$-formula is strictly equivalent to one in which $\mathsf{A}$ occurs at most immediately preceding boxes or atomic formulas—but the fragment consisting of such formulas is simply a no-

[11] We have tacitly corrected a couple of typographical errors in the original. Note that no equivalence numbered "(1)" occurs in Humberstone's paper. The reference is presumably to equivalence (4.11), which reads "$\mathsf{O}\#(\varphi_1, \ldots, \varphi_n) \equiv \#(\mathsf{O}\varphi_1, \ldots, \mathsf{O}\varphi_n)$ for each $n$-ary Boolean connective #."

[12] In personal communication, Humberstone has acknowledged that the objection just discussed is based on an error.

[13] Humberstone actually talks about the quantificational versions of the languages, but this is immaterial to the point at hand.

tational variant of $\mathcal{L}_s^1$ (if one switches $p^s$ for $p$, $p$ for $\mathsf{A}p$, $\Box$ for $\mathsf{A}\Box$, and $\Box^s$ for $\Box$). Humberstone then contends (2004, 48–49):[14]

> It may all the same be worth giving some attention to a potentially disquieting general feature of the above line of thought. As a general rule, it is not a good idea to argue against a more comprehensive language – such as one in which 'S' (or 'A') can operate on an arbitrary formula and not just (in effect) on an atomic formula – and in favor of a less comprehensive one, on the basis of an observation that everything that can be said in the richer language has an equivalent in the poorer, since if we work only with the poorer language, we can no longer formulate the observation in question.[15]

While we agree that there is no purely general argument in favor of "less comprehensive" languages and against "more comprehensive" ones (though the terminology strikes us as tendentious—one might just as well mark the distinction as one between "succinct" and "prolix" languages), Humberstone's particular reason to be worried about such a preference seems misplaced. After all, the proper place for recording such equivalence observations would seem to be a metalanguage common to the object languages in question. Otherwise we should be compelled to include, in the object language of ordinary, non-modal propositional logic, all the infinitely many connectives corresponding to finitary truth functions, in order to be able to formulate all possible truth-functional equivalences, such as that of $p|q$ with $\neg p \vee \neg q$. However, it seems clear that we are *not* so compelled.[16]

Perhaps we can read Humberstone somewhat more charitably as being concerned with the more specific case of languages employing syntactic devices enclosing arbitrarily complex formulas in their scopes (such as $\mathsf{A}$) as opposed to languages with markers that can only be applied to particular primitive symbols (such as the subjunctive marker). But even then the worry seems misplaced. If the reasoning underlying it were sound, we would, for instance, be compelled to formulate ordinary first-order languages with Montagovian generalized quantifiers in addition to the individual constants from which they are derived, since we would otherwise be unable to formulate, in the object language, the equivalences required to see the expressive equivalence of the two approaches. More precisely, the idea would be that in addition to any constant $c$, first-order languages also use

[14] It might be noted that Humberstone's objection is cited approvingly by French (2015, p. 240).

[15] Humberstone here includes a footnote that reads: "A similar point is made in Smiley (1996) with regard to negation." We have tacitly corrected a typographical error in the original.

[16] This consideration would seem to apply equally to Smiley's original version of the argument.

unary variable-binding operators $\mathsf{C}$ whose semantics is given by the stipulation that, in any model for the original language, $\mathsf{C}x\phi$ is satisfied by an assignment $\alpha$ just in case $\phi$ is satisfied by the $x$-variant of $\alpha$ that maps $x$ to the interpretation of the constant $c$ in the model. We can then formulate, in the object language, the observation that either the constant or the corresponding quantifier is dispensible, because the schema

$$\mathsf{C}x\phi \leftrightarrow \phi_x[c]$$

is valid, where $\phi_x[c]$ is the result of replacing the free occurrences of $x$ in $\phi$ with the constant $c$. But surely this is an observation appropriately made in the metalanguage, and there is no principled reason to prefer first-order languages containing both individual constants and their associated generalized quantifiers over the familiar versions (which is not to say, of course, that in *some* cases such a syntactic set-up might not be desirable).

Another issue that has received some attention in the literature (see e.g. French 2013) is the sense in which $\mathcal{L}^1_s$ and $\mathcal{L}^1_{\mathsf{A}}$ are, or are not, notational variants of each other. The translation lemmas 1 and 3, taken together, might seem to provide evidence in favor of the claim that these languages are indeed notational variants. According to Wehmeier (2004, 2005), however, this is not the whole story, for lemmas 1 and 3 make reference only to the notion of truth *at a world* in a model, whereas the notion of translation relevant for assessing notational variance is truth *simpliciter* (relative only to a model). The mentioned results therefore have no immediate bearing on the matter.

As mentioned briefly in §2, truth in a model for $\mathcal{L}^1_{\mathsf{A}}$ is typically taken to be *real-world truth*, that is, we take an arbitrary $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$ to be true in a model $\mathbf{M} = (W, @, R, \mathcal{V})$ just in case $\phi$ is true at the actual world @ of $\mathbf{M}$, $\mathbf{M} \Vdash_@ \phi$. In particular, then, any $\mathcal{L}^1_{\mathsf{A}}$-formula is eligible for truth and falsity in $\mathbf{M}$. Indeed, if we want a notion of truth in a model that applies to any $\mathcal{L}^1_{\mathsf{A}}$-formula whatsoever, and we want to stick to the standard notion of a model, we have no choice in the matter: To get rid of the world argument place in the relation "$\phi$ is true at $w$ in $\mathbf{M}$" we can either plug it with a constant provided by the model (the only choice here is the actual world @) or quantify it away. But neither universal nor existential quantification over worlds produces acceptable results: If we were to call $\phi$ true in $\mathbf{M}$ if and only if $\phi$ is true at *every* world of $\mathbf{M}$, we would end up with true disjunctions (e.g. $p \vee \neg p$) neither of whose disjuncts is true (as long as the model in question contains worlds in which $p$ is true, and worlds in which $p$ is false), and if we opted instead for the existential definition, we would be stuck with false conjunctions (e.g. $p \wedge \neg p$) both whose conjuncts are true. To reiterate: As long as we want truth in a model to apply to arbitrary $\mathcal{L}^1_{\mathsf{A}}$-formulas, and we want to work with the standard notion of a possible-world model, we have no choice but to adopt real-world truth. Both requirements are standard among philosophical practitioners of modal logic.

In Wehmeier's semantics for $\mathcal{L}^1_s$, by contrast, truth in a model is defined only for *subjunctively closed* formulas, that is, $\mathcal{L}^1_s$-formulas in which every occurrence of a subjunctivized sentence letter or a subjunctivized box lies within the scope of a box. The motivation for this restriction is threefold: First, the subjunctive marker behaves much like a world variable that can be bound by a box *qua* world quantifier. Thus, if a subjunctive marker occurs free (i.e. outside of the scopes of any boxes), it corresponds to a free variable in ordinary predicate logic, making the formula in which it occurs free an open formula, and we standardly define truth in a model for first-order languages only for closed formulas, a.k.a. sentences. Second, subjunctive sentence letters stand in for subjunctive phrases of English, and these are likewise not eligible for truth and falsity by themselves: "Aristotle would have become a carpenter" has no truth value; an utterance of the phrase, absent a salient counterfactual situation, would only elicit the response, "Would have become a carpenter *if what*?" Third, as already mentioned in §2, subjunctively closed formulas do in fact have truth values independently of any particular possible world, much like the sentences of predicate logic have truth values independently of any particular variable assignment: We can simply say that a subjunctively closed formula $\phi$ is true in $\mathbf{M}$ if and only if it is true at some world or other in $\mathbf{M}$, or equivalently, if and only if it is true at all worlds in $\mathbf{M}$, or yet again equivalently, if and only if it is true at the actual world of $\mathbf{M}$.[17]

Now it is certainly worth discussing Wehmeier's motivations for restricting truth *tout court* to subjunctively closed formulas; however, since these motivations are not French's targets, we simply note them here. What French objects to is Wehmeier's observation that, if one requires a *proper* translation to map truth-eligible formulas to truth-eligible formulas, $\mathcal{L}^1_s$ and $\mathcal{L}^1_{\mathsf{A}}$ no longer seem to be mere notational variants, despite the fact that the subjunctively closed fragment of $\mathcal{L}^1_s$ is still expressively equivalent to the full language $\mathcal{L}^1_{\mathsf{A}}$, equipped with real-world truth. For while $t_{s,\mathsf{A}}$ does indeed map each subjunctively closed formula of $\mathcal{L}^1_s$ to a truth-eligible formula of $\mathcal{L}^1_{\mathsf{A}}$ (which is not difficult, given that all of $\mathcal{L}^1_{\mathsf{A}}$ is truth-eligible), the reverse translation $t_{\mathsf{A},s}$ maps the $\mathcal{L}^1_{\mathsf{A}}$-truth-eligible formula $p$ to

[17] It is not entirely clear to us whether French (2013) fully appreciates this independence fact, for although he does acknowledge it in his Lemma 1 (p. 1691), he typically speaks of $\mathcal{L}^1_s$ as being endowed with real-world truth, which strikes us as misleading.

the non-subjunctively closed $\mathcal{L}^1_s$-formula $p^s$. As pointed out in §2, in order to translate $\mathcal{L}^1_{\mathsf{A}}$ into the subjunctively closed fragment of $\mathcal{L}^1_s$, we need to follow up $t_{\mathsf{A},s}$ with an operation (to wit, $\mathsf{act}$) that erases the subjunctive markers attached to any sentence letters or boxes that do not lie within the scope of a box. This means that the $\mathcal{L}^1_{\mathsf{A}}$-formulas $p$ and $\mathsf{A}p$, for example, will both be mapped to the single $\mathcal{L}^1_s$-formula $p$. In $\mathcal{L}^1_{\mathsf{A}}$, however, $p$ and $\mathsf{A}p$ have distinct modal profiles (the former is generally contingent, the latter either necessarily true or necessarily false), so given that they both translate to the $\mathcal{L}^1_s$-sentence $p$, it seems that this translation is not a trivial rewriting of one notation into another.

What, then, are French's objections? Essentially, he argues that the combination of Wehmeier's two observations—i.e. that (a) $\mathcal{L}^1_{\mathsf{A}}$ is expressively equivalent to the subjunctively closed fragment of $\mathcal{L}^1_s$ and (b) that $\mathcal{L}^1_{\mathsf{A}}$ is not a trivial notational variant of this fragment of $\mathcal{L}^1_s$—no longer holds if one gives up one or both of the standard requirements for a notion of truth in a model for $\mathcal{L}^1_{\mathsf{A}}$.

In particular, in order to undermine (b), one could jettison the requirement that all $\mathcal{L}^1_{\mathsf{A}}$-formulas be truth-eligible, and restrict truth-eligibility to those $\mathcal{L}^1_{\mathsf{A}}$-formulas that have their truth values independently of a world of evaluation, much like the subjunctively closed $\mathcal{L}^1_s$-formulas. This can of course be done; as French notes, all one needs to do is to restrict truth in a model to those $\mathcal{L}^1_{\mathsf{A}}$-formulas in which every box and every sentence letter occurs within the scope of an $\mathsf{A}$. But of course it was precisely Wehmeier's point to draw attention to the fact that this is *not* how *anyone* had ever defined truth in a model for $\mathcal{L}^1_{\mathsf{A}}$; the main thrust of Wehmeier (2004, 2005) is to show that the persuasiveness of Kripke's modal argument against the description theory of proper names rests precisely on the received assumption that any arbitrary modal formula is truth-eligible. Once the proponent of $\mathcal{L}^1_{\mathsf{A}}$ concedes that truth-eligibility must be restricted in the way suggested by Wehmeier, the languages do indeed become notational variants of sorts—but one has then changed the construal of $\mathcal{L}^1_{\mathsf{A}}$ in a way that has never before been considered in the literature.[18] While the approaches become much more similar once the actuality theorist revises her notion of truth in a model as here suggested, considerations concerning cross-world predication (see below) would still favor the subjunctive marker framework in terms of versatility.

French's second strategy to undermine the combination of Wehmeier's observations is to change the notion of a model and thereby attack observation

[18] A solitary exception is French (2012), who can be read as endorsing a restriction on truth-eligibility in $\mathcal{L}^1_{\mathsf{A}}$ that corresponds to Wehmeier's subjunctive closure; but this paper itself draws some of its inspiration from Wehmeier (2004).

(a) above. The proposal is to consider, instead of the familiar models $\mathbf{M} = (W, @, R, \mathcal{V})$, so-called *d-models* that contain an additional world parameter, the *designated* or *d*-world. Thus such d-models are quintuples $\mathbf{\Delta} = (W, @, w^*, R, \mathcal{V})$ such that $(W, @, R, \mathcal{V})$ is an ordinary model, and $w^*$ is a member of $W$ that may or may not be identical to @.[19]

French (2013) now proposes the following. Define truth at a world in a d-model $\mathbf{\Delta} = (W, @, w^*, R, \mathcal{V})$ for $\mathcal{L}^1_{\mathsf{A}}$-formulas in exactly the same way we did for ordinary models (so that, in particular, the clause for the actuality operator $\mathsf{A}$ makes reference to @, not to $w^*$). Stipulate that an $\mathcal{L}^1_{\mathsf{A}}$-formula $\phi$ is true in $\mathbf{\Delta}$ just in case it is true at $w^*$ in $\mathbf{\Delta}$ (call this d-truth). Further define truth at a world in a d-model for $\mathcal{L}^1_s$ exactly as we did for ordinary models. Define the notion of subjunctive closure in $\mathcal{L}^1_s$ exactly as before. Then stipulate that truth in a d-model for $\mathcal{L}^1_s$ applies only to subjunctively closed $\mathcal{L}^1_s$-formulas, and that such a subjunctively closed $\mathcal{L}^1_s$-formula is true in $\mathbf{\Delta}$ just in case it is true at its designated world $w^*$.[20]

French then shows that, on these definitions, $\mathcal{L}^1_{\mathsf{A}}$ and the subjunctively closed fragment of $\mathcal{L}^1_s$ are not expressively equivalent over d-models; indeed the expressive power of $\mathcal{L}^1_{\mathsf{A}}$ is greater. He concludes that the focus on subjunctively closed formulas in $\mathcal{L}^1_s$ is either misguided—in which case we're back to seeing $\mathcal{L}^1_{\mathsf{A}}$ and $\mathcal{L}^1_s$ as mere notational variants—or a serious defect of the subjunctive marker approach, as it leads to expressive deficits.

But French's analysis here is not convincing, because he has stacked the deck against $\mathcal{L}^1_s$. Let us note first that the inequivalence result he proves is hardly surprising: By endowing $\mathcal{L}^1_{\mathsf{A}}$ with d-truth, we're at the same time giving the language the capacity to make certain statements about the designated world $w^*$; for example, the sentence letter $p$, considered as an $\mathcal{L}^1_{\mathsf{A}}$-formula, being evaluated at $w^*$ by default, then says about the d-model that $\mathcal{V}_{w^*}(p) = \mathsf{T}$. Obviously the subjunctively closed formulas of $\mathcal{L}^1_s$ have no such capacity, for their unmarked sentence letters only ever refer to @, and their subjunctive sentence letters are all bound to modal operators and thus cannot refer to any particular worlds.

However, the expressive capacity of the actuality language has now been extended, through d-truth, in a way we already know to be unsatisfactory (at least once we include quantification in the modal language, i.e. once we move on to $L^1_{\mathsf{A}}$).

[19]Though he didn't call them that, d-models were essentially introduced by Hanson (2006). It has remained unclear what the conceptual benefit of d-models might be.

[20]French also considers a variant definition on which subjunctively closed formulas are considered true in $\mathbf{\Delta}$ just in case they are true at @, but the objections to his argument are in each case the same, so we will not consider this variant separately.

Recall that in quantified modal logic, the whole rationale behind the introduction of the A-operator was that, in its absence, real-world truth gave the language an expressive imbalance. While it can express the truth condition *for everything that is F in @ there is a world w such that it is G in w*, it is unable to express the truth condition *there is a w such that everything that is F in @ is G in w* (i.e. the meaning of an English sentence like *Under certain circumstances, everyone who is rich would have been poor*[21]). Introducing another designated world, $w^*$, into models for the quantified version $L^1_{\mathsf{A}}$ of $\mathcal{L}^1_{\mathsf{A}}$, together with the stipulation that truth in a model be d-truth, creates an expressive imbalance of the same nature, for it will now be impossible to express that there exists a world $w$ such that everything that is $F$ in $w^*$ is $G$ in $w$, even though it *is* possible to express *for everything that is F in* $w^*$ *there is a world w such that it is G in w*. That is, by adding structure to the models we have created a new imbalance between the things we ought to be able to say and the things we *can* say in the extant language. Thus the extension of the notion of a model must go along with an expansion of the modal language in order to correct this imbalance, just as when we extend the notion of a model by including an actual world and simultaneously expand the modal language by including the actuality operator A.

How should we expand the expressive power of quantified $\mathcal{L}^1_{\mathsf{A}}$ for this purpose? Naturally by introducing another operator, say D, which operates just like A except that, instead of pointing to the actual world @, it points to the designated world $w^*$. Then we can express the truth condition in question by means of the modal formula $\Diamond\forall x(\mathsf{D}Fx \rightarrow Gx)$.[22] But now it becomes quite plain that a corresponding expansion of quantified $\mathcal{L}^1_s$ needs to be made before we can fairly compare the operator and the mood marker approaches. The obvious modification of the subjunctive modal language consists in adding another mood marker, $d$, that can annotate modal operators and predicate symbols or, in the propositional case, sentence letters. An atomic formula of the form $p^d$ will then be counted as true at a world $w$ just in case $V_{w^*}(p) = \mathsf{T}$, and we will naturally continue to count formulas in the subjunctive language as subjunctively closed in the requisite sense just in case every sentence letter and every box annotated with the subjunctive marker $s$ lies within the scope of a box. But now, as is easily verified, the expressive equivalence of the subjunctively closed fragment of the subjunctive language, so expanded, again coincides with the expressive power of the actuality language,

[21]Cf. Hazen 1976, Hodes 1984, Wehmeier 2003, Kocurek 2015.

[22]Or rather, in varying-domain models, $\Diamond\forall^{\mathsf{D}}x(\mathsf{D}Fx \rightarrow Gx)$; but we will ignore this nicety.

expanded by the operator $\mathsf{D}$, relative to d-models.[23]

Finally, French (2015) purports to show that “the expressive equivalence of the [predicate marker] and [actuality operator] frameworks is only an artefact of comparing the two different linguistic frameworks in a particular logical setting—namely first-order $\mathsf{S5}$ with the standard modal operators $\Box$ and $\Diamond$” (p. 241). We have, in fact, already shown in §2 that this is not the case, and French actually acknowledges this fact in his Theorem 8.4 (2015, p. 260). So what is the basis for his claim to the contrary?

Well, for the most part, he attacks two systems we don’t consider reasonable instances of the subjunctive marker approach, namely (i) $\mathcal{L}_s^1$ without indicative boxes and (ii) $\mathcal{L}_s^1$ without subjunctive boxes. Neither of these systems is, unsurprisingly, expressively equivalent to $\mathcal{L}_{\mathsf{A}}^1$. To be fair, French eventually comes around to acknowledging the possibility of setting the subjunctive language up in the way we have done in §2 of this paper, but he suggests that there is something dubious about making the indicative–subjunctive distinction not just for predicates and quantifiers (or sentence letters, in the propositional case), but also for boxes.

As he puts it, making this distinction puts proponents of the subjunctive marker approach in the “unsatisfying position of having to claim that there is a hidden ambiguity between indicative and subjunctive versions of the modal operators” (2015, p. 260; we have tacitly corrected a typographical error in the original), and that they end up “having to posit two senses of possibility—an indicative and a subjunctive one” (2015, p. 261).

But of course, if we assume that the possible worlds are structured by an accessibility relation that is not universal, there really are two such senses of possibility, namely possibility in the sense of what *is* possible (i.e. what is true in worlds accessible from the actual world) and possibility in the sense of what *would* have been possible were we in a counterfactual world $w$ (i.e. what is true in worlds accessible from $w$). Thus it is imperative that a formal language designed for talk about possibilities be able to mark this distinction between real and counterfactual possibility. This is precisely what our language $\mathcal{L}_s^1$, with its naked and its subjunctivized modal operators, accomplishes, and in a natural and fully transparent way at that.[24]

[23] For a related argument, see (Wehmeier 2014).

[24] The distinction is of course also present in $\mathcal{L}_{\mathsf{A}}^1$, to wit, in the contrast between $\mathsf{A}\Diamond\phi$ and $\Diamond\phi$, but it is somewhat obscured by the convention to apply real-world truth to the language, so that $\Diamond\phi$, when unembedded, has the same truth conditions as $\mathsf{A}\Diamond\phi$. In $\mathcal{L}_s^1$, by contrast, with the notion of truth in a model restricted to subjunctively closed formulas, the distinction is clearly marked between $\Diamond\phi$ and $\Diamond^s\phi$, with only the former being truth-eligible as a self-standing formula, and the

Moreover, having both indicative and subjunctive versions of the boxes, quantifiers, and predicate symbols is entirely in keeping with the key idea behind the subjunctive language, namely to mark modal distinctions exclusively by means of the interaction of modal operators with mood distinctions at the level of primitive predicative expressions. For just as the predicate symbols are predicative with respect to individuals, so are the quantifiers with respect to sets of individuals (after all, quantifiers are higher-order predicates), and likewise the boxes with respect to sets of possible worlds (after all, modal operators are quantifiers over possible worlds, i.e. higher-order predicates). Indeed, it would seem that we can make distinctions of mood with respect to at least some of the natural-language counterparts of predicate symbols (*ran* vs. *would have run*), quantifiers (*there is* vs. *there would have been*), and modal operators (*it is possible that* vs. *it would have been possible that*) in exactly parallel ways.

What French does not note, but what in fact constitutes something of an embarrassment for the actuality operator approach, is that the quantified version of $\mathcal{L}^1_{\mathsf{A}}$ itself is under some pressure to make use of a mood marker rather than an operator, namely in the actuality quantifiers $\forall^{\mathsf{A}}$ and $\exists^{\mathsf{A}}$ (originally due to Hazen 1990). To see why, consider again the sentence

*Under certain circumstances, everyone who is rich would have been poor.*

The natural first shot at formalization, using the $\mathsf{A}$-operator, is

$$\Diamond\forall x\,(\mathsf{A}Rx \to Px)\,,$$

but a moment's thought shows that this won't do (as pointed out by Bricker 1989 as well as Fara and Williamson 2005). For this sentence might be true simply because there is a possible world $w$ in which none of those who are rich in the actual world exist. The standard fix is to annotate the universal quantifier in the scope of the diamond with $\mathsf{A}$ to make its bound variable range over the domain of the actual world:

$$\Diamond\forall^{\mathsf{A}}x\,(\mathsf{A}Rx \to Px)\,.$$

But note how this device in effect destroys the actuality theorist's commitment to expressing all modal distinctions exclusively through the scopes of operators: The superscript $\mathsf{A}$ on the universal actuality quantifier $\forall^{\mathsf{A}}$ is simply an indicative marker, the dual of our subjunctive marker.

It is of course not strictly necessary to notate the actuality quantifiers in this particular way, i.e. by superscripting a naked quantifier. One might, for example,

latter only being able to occur embedded under a modal.

simply write $\Pi$ instead of $\forall^{\mathsf{A}}$ and continue to use $\forall$ for the world-bound universal quantifier. In this way one avoids the appearance of using a mood marker. However, such a notational choice arguably obscures the fact that there exists an intrinsic semantic relationship between the quantifiers $\Pi$ and $\forall$ in that they both express notions of universal quantification over objects, and the former is the specification of the latter to the domain of the actual world. Even setting actual typographical convention aside, it therefore seems fair to us to impute to the actuality theorist a need to make use of mood markers on quantifiers.

Thus even if one wanted at all costs to stick to scope-bearing devices for the expression of modal distinctions, there are strong reasons to adopt a mood marker anyway. In terms of methodological purity, it thus seems much more consistent to adopt a predicate marker approach from the start.[25]

We note, finally, that the formalization of cross-world predication, as in *John might have been taller than Mary is* or *Under certain circumstances, everyone would have been as generous as they might have been*, poses challenges to the proponent of modal languages based on $\mathsf{A}$ that subjunctive theorists take in stride.[26] Since sentential operators, such as $\mathsf{A}$ and the indexed operators $\mathsf{A}_i$, always include entire formulas within their scopes, they cannot distinguish between, as it were, the two copulas of a single cross-world predicate: *John might have been taller than he is* can neither be formalized as $\Diamond \mathsf{A} T(j, j)$, nor as $\mathsf{A} \Diamond T(j, j)$, and there simply is no other place for $\mathsf{A}$ to go.[27]

Indeed the only recourse open to actuality theorists vis-à-vis cross-world comparatives is to express them via the associated graded positives and an ontology of degrees: Thus *John might have been taller than he is* is paraphrased as something like *There is a height that John has and which is such that John might have had a height greater than it*. This we can indeed formalize in $L^1_{\mathsf{A}}$, but only at the cost of including degree (here: height) talk in the object language, and including the degrees themselves in the structure of our modal models. But even if cross-world comparisons have now become expressible in principle, albeit at some cost, there

[25] That is not to say that one couldn't amend quantified $\mathcal{L}^1_{\mathsf{A}}$ in some other way to make the sentence in question expressible, e.g. by adding Vlach operators (thanks to an anonymous referee for the suggestion). Discussion of such an approach would lead us too far afield; suffice it to say that the actuality quantifiers appear to be the standard remedy, and that, in our view at least, Vlach operators do not have natural counterparts in natural language modal discourse.

[26] For a full discussion of the issue, we refer the reader to (Wehmeier 2012). For background see also (Kemp 2000).

[27] Obviously $\Diamond T(j, j)$ would not do either, since it would be true only if there is a world $w$ such that John's height in $w$ is greater than John's height in $w$, which there isn't.

remains the problem that we cannot distinguish, in the object language, between the contents of the statements *John might have been taller than he is*, on the one hand, and *There is a height that John has and which is such that John might have had a height greater than it*, on the other; the actuality theorist must take these to be literally the same statement.

For proponents of the mood marker approach, by contrast, cross-world predication poses no new challenges at all: Where $R$ is a binary predicate symbol that is to be interpreted as a cross-world relation such as *x is taller than y is*, we simply allow each of the two argument places to be individually adorned by mood markers.[28] In other words, we will express the sentence *John is taller than Mary (is)* as $R^{i,i}(j,m)$, the sentence *Under certain circumstances, John would have been taller than Mary (would have been)* as $\Diamond R^{s,s}(j,m)$, and the sentence *Under certain circumstances, John would have been taller than Mary is* as $\Diamond R^{s,i}(j,m)$.[29] As shown in (Wehmeier 2012), the requisite changes to the model-theoretic apparatus are rather minimal; we just need to allow cross-world relations, i.e. relations that may take arguments from distinct worlds. Such relations are clearly definable in the more traditional languages making use of degrees as discussed above, so there is no additional commitment incurred by the mood marker approach. Indeed, it is easy to incorporate a degree-based analysis into the subjunctive marker-language itself, so that, in this framework, it is finally possible to represent as distinct the English sentences *John might have been taller than he is* on the one hand, and *There is a height that John has and which is such that John might have had a height greater than it*, on the other, to wit, as $\Diamond R^{s,i}(j,j)$ and $\exists h(T(j,h) \wedge \Diamond\exists h'(T(j,h') \wedge h' > h))$, respectively.

Indeed, if we set aside the linguistic question of how best to formalize cross-world predications as they occur in natural languages, it turns out that the mood marker approach has a much greater versatility than actuality-operator based approaches when it comes to logical generalizations of the framework, for there is no limit to the number of argument places of a relation that we might adorn with mood markers, so that we can easily talk about relations that reach across more than just two worlds. Thus we could, for example, consider ternary relations that hold across worlds, and form expressions like $\Diamond_1\forall^1 x\,\Box_2\exists^2 y\,\Diamond_3\forall^3 z\,R^{2,1,3}(x,y,z)$

[28]It is notationally more elegant in this setting to assign indicative predicates an explicit indicative marker rather than leaving them naked as in the simple modal environment, but this is evidently a trivial notational change.

[29]We are here using "$i$" as an indicative marker. Note that, in definition 5, we used "$i$" as a variable over natural numbers *qua* subjunctive markers. We trust that this does not cause any confusion.

whose truth condition is that there exists a world $w_1$ such that for all objects $a$ existing at $w_1$ and all worlds $w_2$, there exists an object $b$ at $w_2$ and a world $w_3$ such that every object $c$ existing at $w_3$ is such that $R$ holds between $a$ as it is in $w_2$, $b$ as it is in $w_1$, and $c$ as it is in $w_3$. This is not something we can simulate by means of degrees, since degrees arise only from comparative and equivalence, and hence binary, relations.

It therefore appears that, while the mood marker languages can do everything actuality operator languages can do, in the case of cross-world predication mood markers properly outperform operator languages. There is thus not only no reason to neglect mood markers in favor of scope-bearing operators, but indeed substantive reason to prefer the mood marker-approach to the actuality operator approach.

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