TRACTARIAN FIRST-ORDER LOGIC: IDENTITY AND THE N-OPERATOR

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> As it is obviously easy to express how propositions can be constructed by means of this operation and how propositions are not to be constructed by means of it, this must be capable of exact expression.
>
> 5.503

1. Introduction

Wittgenstein introduces two major proposals concerning logical notation in the *Tractatus*. One is that every proposition be represented in a notation that uses only one logical operator, which he calls the N-operator:

> [...] every proposition is the result of successive applications of the operator N'$(\bar{\xi})$ to the elementary propositions. (6.001)[1]

The other is that the sign for identity be eliminated from logical symbolism:

> Identity of the object I express by identity of the sign and not by means of a sign of identity. Difference of the objects by difference of the signs. (5.53)
>
> The identity sign is therefore not an essential constituent of logical notation. (5.533)

While Wittgenstein illustrates the application of each of these conventions with a handful of examples, neither proposal is developed systematically, leaving both open to interpretation. Indeed, as Carnap (1937, p. 50) laments, "since Wittgenstein does not formulate any new rule of substitution but only states a number of examples, it is not clear how he intends to carry out his method". This has resulted in flourishing literatures on the interpretation and evaluation of both proposals.

Skepticism about the viability of Wittgenstein's notational innovations is common. Fogelin (1987, p. 78) argues that the N-operator cannot be used to generate multiply mixed general propositions, and thus constitutes "a fundamental error in the logic of the *Tractatus*". Judgments on Wittgenstein's proposal for eliminating the identity sign have been equally dire: Black (1964, p. 293) finds it "doubtful whether his innovations would work in practice", and Carnap (1937, p. 50) argues that "a closer examination shows that his method of variables leads to certain difficulties". Landini (2007, p. 256) claims that Wittgenstein's convention "encounters serious difficulties for forming a deductive system". One purpose of this essay is to dispel such doubts. To this end, we show that Wittgenstein's notational conventions can be systematically implemented in an expressively adequate system of first-order logic for which sound and complete proof procedures are available.

Surprisingly, little effort has been invested in the *simultaneous* implementation of Wittgenstein's two proposals. White (2006, p. 97) merely suggests that the conventions could be combined. Schroeder (2006, p. 83 fn. 31) and Floyd (2002, pp. 342-345) provide some examples of propositions in a notation employing both conventions, but do not offer a systematic treatment and, in particular, no translation rules with respect to standard notation. McGray (2006) has the most in-depth discussion of the simultaneous application of both proposals, but he argues that no systematic rules of translation can be given: "Elimination of the sign of identity from genuine propositions will require careful analysis on a case-by-case basis. [...] there is no procedural way of transcending the individual case analysis" (ibid., p. 161). As we show, it is entirely possible to implement both conventions simultaneously in a systematic and transparent manner.

Other readers have attempted to systematically apply one or the other of these Tractarian proposals. Some of the resulting logical systems, however, only function under certain presuppositions about the size of the domain of objects. For example, Landini's (2007, pp. 263-264) interpretation of the identity convention requires an infinite domain, while characterizations of the N-operator by Anscombe (1959) and Glock (1996) require a domain of finite size.[2] Yet in the *Tractatus* Wittgenstein leaves open the question of the size of the domain, entertaining the possibility that it might be infinite (4.2211, 5.535), but insisting that this is not a question that can be decided by logic alone:

> [...] we cannot give the number of names with different meanings [...] (5.55)

> [...] it is senseless to speak of the *number of all objects.* (4.1272)

Thus, by imposing constraints on the size of the domain, these interpretations run counter to a major Tractarian tenet. A reading that does not require a choice of domain size, and hence is consistent with these passages of the *Tractatus*, therefore seems desirable. We provide just such a reading.

In the following we first deal with Wittgenstein's proposals individually, and then combine them into a single system. In section 2 we distinguish and discuss interpretations of Wittgenstein's procedure for eliminating the identity sign, and provide rules for the systematic translation of formulas between three species of identity-free logic and standard first-order logic with identity ($\text{FOL}^=$). In section 3 we present the syntax and semantics of a language that has as its only logical operator Wittgenstein's N, and further provide rules for the systematic translation of formulas between this language and $\text{FOL}^=$. A unified language of Tractarian first-order logic incorporating both of Wittgenstein's conventions is then presented and shown to be expressively equivalent to $\text{FOL}^=$ in section 4. We finally develop, in section 5, sound and complete Tractarian tableau calculi for (a) the case where the size of the domain is allowed to vary arbitrarily, (b) the case where the size of the domain is infinite, and (c) the case where the domain is of a fixed finite cardinality n. In the conclusion, we evaluate arguments to the effect that Wittgenstein had no interest in the formulation of a precise system of logic. We also discuss his contention that N is the *sole* logical operator of his Tractarian language.

2. Identity

In a pioneering paper, Hintikka (1956) explicitly formulated two possible ways of interpreting Wittgenstein's 5.53 directive to express identity of object through identity of sign and difference of objects through difference of signs. Hintikka pointed out that his "strongly exclusive" interpretation is incompatible with one of the examples in the *Tractatus* and concludes that Wittgenstein's intended convention coincides with his own "weakly exclusive" interpretation. Unfortunately, Hintikka's accomplishment has not been sufficiently appreciated due to persistent mischaracterizations of his analysis in the literature. In this section, we clarify and expand on Hintikka's two interpretations, identify a variant of the strongly exclusive one that *is* compatible with the *Tractatus*, and provide translation rules between these exclusive readings and $\text{FOL}^=$. We then call upon posthumously published work of

Ramsey's to further support Hintikka's identification of the weakly exclusive reading with Wittgenstein's own intended convention. Finally, we address several confusions and misunderstandings in the literature regarding Wittgenstein's exclusive interpretation of the variables.

2.1. **Three Interpretations of the 5.53 Convention.** Wittgenstein's pronouncement at 5.53 is purely programmatic, and does not tell us exactly how he proposes to express identity and difference of objects in practice. The examples provided in 5.531 through 5.5321 go some way to explaining the underlying idea.[3]

> I write therefore not "$f(a,b).\, a = b$", but "$f(a,a)$" (or "$f(b,b)$"). And not "$f(a,b).\, {\sim} a = b$", but "$f(a,b)$". (5.531)
>
> And analogously: not "$(\exists x,y).f(x,y).\, x = y$", but "$(\exists x).f(x,x)$"; and not "$(\exists x,y).f(x,y).\, {\sim} x = y$", but "$(\exists x,y).f(x,y)$".
> (Therefore instead of Russell's "$(\exists x,y).f(x,y)$":
> "$(\exists x,y).f(x,y) \,.\!\vee\!.\, (\exists x).f(x,x)$".) (5.532)
>
> Instead of "$(x)\colon fx \supset x = a$" we therefore write e.g. "$(\exists x).fx \,.\!\supset\!.\, fa : {\sim}(\exists x,y).fx.fy$".
> And the proposition "*only* one x satisfies $f(\;)$" reads: "$(\exists x).fx : {\sim}(\exists x,y).fx.fy$". (5.5321)

The second translation in 5.531 tells us that Wittgenstein's $f(a,b)$ presupposes, by employing the distinct free variables a and b, that these variables are assigned distinct values. Such a presupposition is naturally understood as requiring that a variable assignment satisfying a given formula must be 1-1 on the free variables occurring in that formula. We call this the *Satisfaction Principle*: A formula ϕ can only be satisfied by variable assignments σ that are 1-1 on the variables occurring free in ϕ.[4]

The first translation in 5.532 is not at all surprising, for the proposition $(\exists x,y).f(x,y).\, x = y$ is equivalent to $(\exists x).f(x,x)$ in $\mathrm{FOL}^{=}$, and so the latter is quite obviously the expression of choice for the former when use of the identity sign is not permitted. But the second example is of great interest: When Wittgenstein writes down a formula of the form $(\exists x,y).f(x,y)$, it is to be understood that the values of the variables x and y must not concur. How can this be systematically ensured? We note first that the concurrence prohibition regarding x and y must be due to the overlapping of the corresponding quantifier scopes. In the third translation offered in 5.532, the quantified variable in the second disjunct must range over *all* objects in the domain; its range is thus not constrained by the fact that bound variables occur in the first disjunct. Indeed, a systematic procedure according to which the bound variable in the second disjunct *would* be constrained by those of the first seems impossible, unless one permits such a procedure to violate standard means of scope indication. It thus appears that there are two ways in which scope overlap can systematically determine variable range restrictions.

First, by looking inward (Hintikka's "weakly exclusive" interpretation): In order for $(\exists x, y).f(x, y)$ to be true, we must first find a witness a for the leading existential quantifier $\exists x$. Since no other values have yet been assigned to any variables, there is no way we could exempt a particular object from the range of $\exists x$, so we're forced to let x vary over the entire domain. But $(\exists x, y).f(x, y)$ contains $(\exists y).f(x, y)$, i. e., the scope of $\exists y$, as a subformula. Within this subformula, the variable x (to which we have assigned a as its value) occurs free. If we stipulate generally that the range of a bound variable z contains all elements of the domain except the values of any variables that occur free within the scope of the quantifier binding z, we obtain the following: $(\exists x, y).f(x, y)$ is true if and only if for some object a in the domain, $(\exists y).f(x, y)$ is true when a is assigned to x (no restriction on x, because no variable occurs free within the scope of $\exists x$); further, $(\exists y).f(x, y)$ is true under the assignment of a to x just in case there is an object b other than a such that $f(x, y)$ is true when a is assigned to x and b is assigned to y (because x occurs free within the scope of $\exists y$, the value a of x is excluded from the range of y). Putting things together, we have: $(\exists x, y).f(x, y)$ is true if and only if for some object a and some object b other than a, $f(x, y)$ is true when a is assigned to x and b is assigned to y. These are exactly the truth conditions of the Russellian formula $(\exists x, y).f(x, y).{\sim}x = y$, so we have found a first way of implementing the 5.53 convention in accord with those examples of Wittgenstein's we have considered so far.

The second way proceeds by looking outward (Hintikka's "strongly exclusive" interpretation): As with the weakly exclusive interpretation, we have no means of determining any objects that should be excluded from the range of the leading quantifier $\exists x$. If we stipulate generally that the range of a bound variable z contains all elements of the domain except the values of any variables in whose scope z occurs (where the scope of a bound variable is the scope of the quantifier that binds it), it follows that y cannot take whatever value has been assigned to x. In other words, once we pick a witness a for x, this object a is excluded from the range of y. Again, this results in the formula $(\exists x, y).f(x, y)$

being true just in case there is an object a and an object b other than a such that $f(x, y)$ is true when a is assigned to x and b is assigned to y, and these are just the truth conditions of the Russellian formula $(\exists x, y).f(x, y). \sim x = y$. We have thus found a second way of implementing 5.53 that coheres with Wittgenstein's examples.

We postpone discussion of the examples in 5.5321 for a moment, because some readers will wonder whether there is any difference between the weakly and the strongly exclusive interpretation. Consideration of the formula $\forall y(\forall x Px \supset Py)$ shows that there is. Under the weakly exclusive interpretation, this formula expresses just what it expresses classically (and hence it is a logical truth). This is because no variables occur free within the scopes of y or of x, and so both bound variables vary over all elements of the domain. But under the strongly exclusive interpretation, the formula has the same truth conditions as the classical formula $\forall y(\forall x(x \neq y \supset Px) \supset Py)$, because $\forall x$ occurs within the scope of $\forall y$. This latter formula, however, is falsifiable.

We have said enough about the strongly exclusive reading to enable us to deal with sentences (formulas without free occurrences of variables), but we do not yet have a clear understanding of how we might treat free variables. Hintikka (1956) proposes to take as the scope of a free variable the entire formula within which it occurs free. Let us call the resulting version of the strongly exclusive reading the *broad* strongly exclusive interpretation: From the range of a bound variable z, exclude the values of all variables within whose scopes z occurs (where the scope of a free variable is the entire formula in which it occurs free). The alternative, not considered by Hintikka, is to deny that free variables have scopes at all. The resulting interpretation, which we'll call the *narrow* strongly exclusive reading, can then be stated as follows: From the range of a bound variable z, exclude the values of all variables within whose scopes z occurs (where free variables have no scopes).

The difference between the two strongly exclusive readings is illustrated in the formula $\exists x Px \wedge Pa$. Under the broad version, the free variable a has the entire formula as its scope, so that the bound variable x is to be read exclusively from a. In other words, the formula translates into the Russellian $\exists x(x \neq a \wedge Px) \wedge Pa$. Under the narrow version, by contrast, x does not occur in the scope of any variable and thus has an unrestricted range (which includes the value of a). That is, the narrow strongly exclusive reading interprets the formula in exactly the same way as the standard inclusive reading.[5] Turning now to 5.5321, we see a case in the text where the broad and narrow strongly exclusive readings yield different results.

As Hintikka observed, the first example in 5.5321 is simply incorrect on the broad strongly exclusive reading: The formula Wittgenstein proposes translates, on this reading, into standard notation as

$$(\exists x(x \neq a \wedge Fx) \supset Fa) \wedge \neg\exists x\exists y\,(x \neq a \wedge y \neq a \wedge x \neq y \wedge Fx \wedge Fy)\,,$$

which does *not* say that at most a satisfies F.[6] We thus have textual evidence against the tenability of the broad strongly exclusive reading as an interpretation of Wittgenstein's Tractarian convention. But things are different with respect to the narrow strongly exclusive reading; if the free variable a has no scope, and thus induces no range restrictions, the formula Wittgenstein proposes does indeed have the truth conditions of the Russellian formula it is supposed to translate. We thus have no means to decide, from the text of the *Tractatus*, whether Wittgenstein intended the weakly exclusive or the narrow strongly exclusive reading.[7]

2.2. **The Translation Procedures.** It is time to provide more precise accounts of the three interpretations, and in particular, algorithms for translating between each of the exclusive dialects and FOL$^=$. We assume that the non-logical vocabulary of our language contains an unlimited stock of n-ary relation symbols for each $n \geq 1$, but no function symbols, in keeping with the practice of *Principia Mathematica* (Whitehead and Russell 1910; henceforth cited as *Principia*). Whitehead and Russell were able, by means of the theory of descriptions, to simulate functions (in the now standard sense) by relations (which are *propositional* functions in their sense). For example, a unary function f would be emulated in *Principia* as a binary relation R, where xRy means that x is the value of f for argument y. A function term ft can then be defined to mean $(\iota y)(yRt)$, or in Whitehead and Russell's alternative notation, R‘t. The theory of descriptions is available in each of the exclusive interpretations, for "the F is G" can there be expressed as $\neg\exists x\exists y(Fx \wedge Fy) \wedge \exists x(Fx \wedge Gx)$. Thus Tractarian logic can emulate functions in just the way *Principia* does. We also do not include a separate syntactical category of *names*, or constant symbols, though this is for expository expediency only. For our purposes, we represent names as free variables under a fixed variable assignment.

It is easy to see how to translate from the identity-free language with weakly exclusive variables into FOL$^=$.[8] The translation $\mathcal{R}_w(Rx_1 \ldots x_n)$ of an atomic formula $Rx_1 \ldots x_n$ is just $Rx_1 \ldots x_n$ itself. The translation of the negation of ϕ is the negation of the translation of ϕ, i.e. $\mathcal{R}_w(\neg\phi) = \neg\mathcal{R}_w(\phi)$. Similarly, the translation of a conjunction is the conjunction of the translations of the conjuncts, i.e. $\mathcal{R}_w(\phi \wedge \psi) = \mathcal{R}_w(\phi) \wedge \mathcal{R}_w(\psi)$, and analogously for the other propositional connectives. Finally, if $y_1, \ldots, y_n$ are precisely the variables occurring free in ϕ other than x, we let $\mathcal{R}_w(\exists x\phi)$ be

$$\exists x \left(\neg x = y_1 \wedge \ldots \wedge \neg x = y_n \wedge \mathcal{R}_w(\phi)\right),$$

and $\mathcal{R}_w(\forall x\phi)$ be

$$\forall x \left(\neg x = y_1 \wedge \ldots \wedge \neg x = y_n \supset \mathcal{R}_w(\phi)\right).$$

For the reverse translation $\mathcal{W}$, from FOL$^=$ into the weakly exclusive identity-free language, we proceed as follows: Where $Rx_1 \ldots x_n$ is an atomic formula, but not an equation, its translation $\mathcal{W}(Rx_1 \ldots x_n)$ is just the formula $Rx_1 \ldots x_n$ itself. Now pick a tautology $\top(x)$ in the sole free variable x, say $Px \vee \neg Px$, and let $\mathcal{W}(x = x)$ be this tautology. Pick a contradiction $\bot(x, y)$ in precisely the two free variables x and y, say $Rxy \wedge \neg Rxy$, and let $\mathcal{W}(x = y)$ be this contradiction. Thus equations with the same variable on both sides of the equality sign are translated as tautologies, and those with distinct variables flanking "=" are translated as contradictions.[9] We translate the negation of ϕ as the negation of the translation of ϕ, i.e. $\mathcal{W}(\neg\phi) = \neg\mathcal{W}(\phi)$. The conjunction of ϕ and ψ is translated as the conjunction of the translation of ϕ and the translation of ψ, i.e. $\mathcal{W}(\phi \wedge \psi) = \mathcal{W}(\phi) \wedge \mathcal{W}(\psi)$. The other propositional connectives are treated analogously. Finally, $\mathcal{W}(\exists x\phi)$ is

$$\exists x\mathcal{W}(\phi) \vee \mathcal{W}(\phi_x[y_1]) \vee \ldots \vee \mathcal{W}(\phi_x[y_n]),$$

and $\mathcal{W}(\forall x\phi)$ is

$$\forall x\mathcal{W}(\phi) \wedge \mathcal{W}(\phi_x[y_1]) \wedge \ldots \wedge \mathcal{W}(\phi_x[y_n]),$$

where $y_1, \ldots, y_n$ are precisely the variables occurring free in ϕ other than x.[10]

For both versions of the strongly exclusive interpretation, we need auxiliary binary translation functions $\mathcal{R}_s$ and $\mathcal{S}$, from which we will be able to define the version-specific unary translations $\mathcal{R}_s^b$, $\mathcal{R}_s^n$, $\mathcal{S}_b$, and $\mathcal{S}_n$.

The binary translation function $\mathcal{R}_s$ takes as arguments a strongly exclusive formula ϕ and a finite set V of variables (we think of the members of V as the variables whose values are to be excluded from the ranges of any quantifiers encountered later in the construction tree of the formula). For atomic formulas $Rx_1 \ldots x_n$ and arbitrary V, we let $\mathcal{R}_s(Rx_1 \ldots x_n, V)$ be $Rx_1 \ldots x_n$ itself. $\mathcal{R}_s(\neg\phi, V)$ is $\neg\mathcal{R}_s(\phi, V)$ and

$\mathcal{R}_s(\phi \wedge \psi, V)$ is $\mathcal{R}_s(\phi, V) \wedge \mathcal{R}_s(\psi, V)$; similarly for the other propositional connectives. Finally, $\mathcal{R}_s(\exists x\phi, V)$ is

$$\exists x\,(\neg x = y_1 \wedge \ldots \wedge \neg x = y_n \wedge \mathcal{R}_s(\phi, V \cup \{x\}))\,,$$

and $\mathcal{R}_s(\forall x\phi, V)$ is

$$\forall x\,(\neg x = y_1 \wedge \ldots \wedge \neg x = y_n \supset \mathcal{R}_s(\phi, V \cup \{x\}))\,,$$

where $y_1, \ldots, y_n$ are precisely the elements of V.[11]

The binary translation function $\mathcal{S}$ takes as arguments a standard formula ϕ and a finite set V of variables (we think of the members of V as the variables whose values would mistakenly drop out of consideration in replacing a standard with a strongly exclusive quantifier). Where $Rx_1 \ldots x_n$ is an atomic formula, but not an equation, $\mathcal{S}(Rx_1 \ldots x_n, V)$ is just the formula $Rx_1 \ldots x_n$ itself. Let $\mathcal{S}(x = x, V)$ be $\top(x)$, and where x and y are distinct variables, let $\mathcal{S}(x = y, V)$ be $\bot(x, y)$. $\mathcal{S}(\neg\phi, V)$ is $\neg\mathcal{S}(\phi, V)$ and $\mathcal{S}(\phi \wedge \psi, V)$ is $\mathcal{S}(\phi, V) \wedge \mathcal{S}(\psi, V)$; similarly for the other propositional connectives. Finally, $\mathcal{S}(\exists x\phi, V)$ is

$$\exists x\mathcal{S}(\phi, V \cup \{x\}) \vee \mathcal{S}(\phi_x[y_1], V) \vee \ldots \vee \mathcal{S}(\phi_x[y_n], V),$$

and $\mathcal{S}(\forall x\phi, V)$ is

$$\forall x\mathcal{S}(\phi, V \cup \{x\}) \wedge \mathcal{S}(\phi_x[y_1], V) \wedge \ldots \wedge \mathcal{S}(\phi_x[y_n], V),$$

where $V = \{y_1, \ldots, y_n\}$.[12]

Now for translation into standard notation, the broad and the narrow strongly exclusive interpretations differ only in which variables are written into the set V at the outset of the translation procedure: For the broad strongly exclusive reading, we let $\mathcal{R}^b_s(\phi)$ be $\mathcal{R}_s(\phi, \mathsf{FV}(\phi))$, where $\mathsf{FV}(\phi)$ is the set of variables occurring free in ϕ. In other words, the values of the variables occurring free in the formula to be translated will be excluded from the ranges of any quantifiers subsequently encountered in the translation process. For the narrow strongly exclusive reading, we put $\mathcal{R}^n_s(\phi)$ equal to $\mathcal{R}_s(\phi, \emptyset)$, leaving the ranges of subsequent quantifiers unrestricted by the values of any variables occurring free in ϕ.

It remains to discuss the translation from FOL$^=$ into the two strongly exclusive notations. The situation here is quite analogous: In the broad strongly exclusive case, we must recall the values of any variables free in the original formula, because they must be added in as special cases when a Russellian quantifier is replaced by a strongly exclusive one. Thus we set $\mathcal{S}_b(\phi)$ equal to $\mathcal{S}(\phi, \mathsf{FV}(\phi))$. But in the narrow strongly exclusive case, the values of variables free in the original formula do not constrain the ranges of subsequent strongly exclusive quantifiers, so we need not keep track of them. In other words, we can let $\mathcal{S}_n(\phi)$ be $\mathcal{S}(\phi, \emptyset)$.[13]

For the sake of expediency, we take the $\mathcal{R}$-translations as providing the meanings of formulas in the various exclusive dialects. More precisely, let $\mathcal{U}$ be a model (that is, a non-empty domain U together with an n-ary relation $P^{\mathcal{U}}$ over U for every n-ary relation symbol P of the language) and σ a $\mathcal{U}$-assignment (that is, a function mapping individual variables to elements of U). Where ϕ is a formula of FOL$^=$, we assume familiarity with the notion of ϕ being satisfied in $\mathcal{U}$ by σ in the standard, Tarskian sense; we refer to this notion as t-satisfaction. If ϕ is a formula of one of our identity-free languages, we say that ϕ is w-satisfied (respectively, sn-satisfied and sb-satisfied) in $\mathcal{U}$ by σ just in case $\mathcal{R}_w(\phi)$ (respectively, $\mathcal{R}_s^n(\phi)$ and $\mathcal{R}_s^b(\phi)$) is t-satisfied in $\mathcal{U}$ by σ.[14]

In any of the languages considered, a *sentence* is a formula in which no variables occur free. In a given model $\mathcal{U}$, a sentence ϕ of FOL$^=$ is t-satisfied by some $\mathcal{U}$-assignment if and only if it is t-satisfied by all $\mathcal{U}$-assignments; one thus defines such a sentence ϕ to be t-true in $\mathcal{U}$ just in case either one (and hence both) of these conditions obtains. Via the various $\mathcal{R}$ translations, the notion of truth can also be defined for sentences of our identity-free languages in analogy with the case of satisfaction: ϕ is w-true (respectively, sn-true and sb-true) in $\mathcal{U}$ just in case $\mathcal{R}_w(\phi)$ (respectively, $\mathcal{R}_s^n(\phi)$ and $\mathcal{R}_s^b(\phi)$) is t-true in $\mathcal{U}$.

With these preparations out of the way, we can now address the question of the expressive adequacy of the various exclusive interpretations of the variables. The answer depends on whether the formulas considered are allowed to contain names (i.e. free variables) and, in case names are present, also on whether different names are permitted to have the same bearer.

Wittgenstein holds that names are not essential for a description of the world:

> One can describe the world completely by completely generalized propositions, i.e. without from the outset co-ordinating any name with a definite object. (5.526)[15]

Now for completely generalized propositions (i.e. sentences in our terminology), one can show that each of the exclusive dialects is expressively equivalent to FOL$^=$ in the following sense:

Lemma 1. For all sentences ϕ of FOL$^=$ and all models $\mathcal{U}$, the following are equivalent:

(1) ϕ is t-true in $\mathcal{U}$

(2) $\mathcal{W}(\phi)$ is w-true in $\mathcal{U}$
(3) $\mathcal{S}_b(\phi)$ is sb-true in $\mathcal{U}$
(4) $\mathcal{S}_n(\phi)$ is sn-true in $\mathcal{U}$.[16]

This result establishes expressive equivalence with FOL$^=$ in the sense that any distinctions between ways the world might be that can be drawn by means of a sentence ϕ in FOL$^=$ (i.e., any way of carving up the possibilities $\mathcal{U}$ into those compatible with ϕ and those incompatible with it) can be drawn just as well with the help of ϕ's translation into any of our exclusive dialects. Thus Wittgenstein's Tractarian variable convention, no matter which interpretation we give it, is entirely successful in providing an identity-free notation that is as powerful as FOL$^=$ when it comes to describing the world.

What if we wish to use a language containing names? Wittgenstein himself requires that no two names corefer. He says as much in 5.53, and we have already seen that the second example of 5.531 also commits him to this constraint. In our setting, where names are represented as free variables, this simply means that variable assignments must be 1-1 on the variables occurring free in a formula, which is what we earlier called the Satisfaction Principle. If we follow Wittgenstein in imposing this requirement, we still have full expressive equivalence with FOL$^=$ even in the presence of names:

Lemma 2. For all formulas ϕ of FOL$^=$, all models $\mathcal{U}$, and all $\mathcal{U}$-assignments σ that are 1-1 on $\mathsf{FV}(\phi)$, the following are equivalent:
(1) ϕ is t-satisfied in $\mathcal{U}$ by σ
(2) $\mathcal{W}(\phi)$ is w-satisfied in $\mathcal{U}$ by σ
(3) $\mathcal{S}_b(\phi)$ is sb-satisfied in $\mathcal{U}$ by σ
(4) $\mathcal{S}_n(\phi)$ is sn-satisfied in $\mathcal{U}$ by σ.[17]

The only setting in which full equivalence with FOL$^=$ does not obtain is when the language is allowed to contain possibly coreferring names; there simply is no way to express, in any of the exclusive dialects, what we can say in FOL$^=$ by means of the formula $a = b$.[18] But it seems inappropriate to hold this against the Tractarian proposal, first because the use of names is not essential to a description of the world, and second because Wittgenstein requires that distinct names never refer to the same object. However, the inexpressibility of $a = b$ in the exclusive dialects brings up a subtle hitch that has gone unnoticed in the literature.

In his first example of 5.531, Wittgenstein proposes to translate the Russellian $f(a, b) . a = b$ as $f(a, a)$ or as $f(b, b)$. Since $f(a, a)$ and $f(b, b)$ are inequivalent under *any* interpretation, it is readily apparent that there is something peculiar about the example. The problem is that, while $f(a, b) \,.\, a = b$ implies both $f(a, a)$ and $f(b, b)$ in FOL$^=$, the implications do not reverse. If the Satisfaction Principle were imposed on the Russellian language, $f(a, b) \,.\, a = b$ should translate into a contradiction, because $a = b$ can then never be true. Russell, however, does not operate under the Satisfaction Principle, which gives rise to an incommensurability between the two languages. The best Wittgenstein can do in *this* situation is to propose as a translation a consequence of Russell's formula that is independent of the Satisfaction Principle—such as $f(a, a)$. But it is clear that neither $f(a, a)$ nor $f(b, b)$ can count as an adequate translation no matter whether the Satisfaction Principle is imposed or not, for in either case they fail to have the same truth conditions as $f(a, b) \,.\, a = b$. As explained above, our translation procedures only pertain to scenarios in which neither language contains distinct but coreferring names, so that this example has no bearing on our translatability results.

2.3. **Translating *Scheinsätze*.** There is some prima facie tension between our providing Tractarian translations for *all* formulas of FOL$^=$ on the one hand, and Wittgenstein's pronouncement at 5.534:

> And we see that apparent propositions [*Scheinsätze*] like: "$a = a$", "$a = b \,.\, b = c \,.\supset a = c$", "$(x) \,.\, x = x$", "$(\exists x) \,.\, x = a$", etc. cannot be written in a correct logical notation at all.

This passage is often interpreted as a claim that the formulas in question utterly fail to express truth conditions. For example, Kremer (2007, p. 155) argues that "supposed propositions like '$(x)x = x$' and '$(\exists x)x = a$' have no counterparts in [Wittgenstein's] notation", because

> [...] for Wittgenstein, neither of these pseudo-propositions corresponds to anything that can be said—not even to something *sinnlos* such as a tautology or a contradiction. (ibid., p. 168 n. 65)[19]

Thus, according to this reading, *Scheinsätze* (rendered as "apparent propositions" at 5.534, but as "pseudo-propositions" at 4.1272, 5.535, and 6.2) fail to correspond to any sayable propositional content; a procedure for eliminating identity must therefore not provide translations of such formulas.

As noted by Black (1964, p. 202), Wittgenstein does not use the term *Scheinsatz* pejoratively to indicate a nonsensical piece of gibberish that has no counterpart in a correct notation.[20] Rather, pseudo-propositions are to be contrasted with "proper" propositions (just as pseudo-concepts and proper concepts are contrasted at 4.1272). This is made clear in the *Notebooks*:

> Pseudo-propositions [*Scheinsätze*] are such as, when analysed, turn out after all only to shew what they were supposed to say. (Wittgenstein 1961a, p. 16)

A proper proposition does not attempt to say what is already shown in a correct symbolism. Wittgenstein evidently thought that the formulas in 5.534 were pseudo-propositions *because* they violated the say-show distinction, for the earliest known draft of his proposal to eliminate the identity sign (1961a, p. 34) is immediately preceded by the remark that became *Tractatus* 4.1212: "What *can* be shown, *cannot* be said." Identities of the form $x = x$ are redundant, because they "say nothing" (5.5303) that isn't already shown by the use of the variable x. Identities of the form $x = y$, where x and y are distinct, attempt to say that the variables x and y are to be assigned the same object—in violation of what is shown, in a correct notation, by the use of distinct variables.

Wittgenstein should be understood at 5.534 as saying that the formulas in question *literally* cannot be written down, because a correct logical notation simply has no sign for identity. While these formulas violate the say-show distinction, they are still systematically associated with definite satisfaction conditions, which can likewise be represented by formulas that are written in a correct notation. For example, in Russell's notation, the sentence $(\exists x, y).\sim x = y$ is true exactly when there are at least two individuals and is thus usually taken to express "there are (at least) two objects". For Wittgenstein this attempt to speak of the number of objects results in a *Scheinsatz* (4.1272). Nevertheless, applying any of the translation functions into exclusive readings described above to this sentence results in a formula equivalent to $(\exists x)(\exists y)(Rxy \vee \sim Rxy)$. There is nothing objectionable about *this* formula from a Wittgensteinian standpoint, and under any exclusive reading of the variables, it is true precisely when there are at least two objects. Furthermore, if the ontology of the *Tractatus* contains at least two objects, the formula is a tautology, and otherwise, it is a contradiction; thus the translation does not say any more than the Russellian formula, because tautologies and contradictions say nothing at all (4.461). Yet the translation *shows*, through its two leading existentially quantified variables, what its Russellian counterpart attempts to *say*. The proposed translations between Russellian and "correct" notation are therefore paradigmatic examples of what Wittgenstein characterizes as the task of philosophy at 4.112, namely, "to make propositions clear" and to "make clear and delimit sharply the thoughts which otherwise are, as it were, opaque and blurred."

In one case, we have sufficient evidence from the text to prove on Tractarian grounds that a Wittgensteinian pseudo-proposition is appropriately translated into a tautology and hence, while *sinnlos*, not nonsense. Let us begin with the Russellian formula $Pa \vee \neg Pa$. Since it does not contain the equality sign, and only one variable occurs in it, the translation is simply $Pa \vee \neg Pa$ itself. Now by 5.441, according to which $(\exists x).fx\,.\,x = a$ says the same as fa, the Russellian formula $Pa \vee \neg Pa$ says the same as $\exists x\,([Px \vee \neg Px] \wedge x = a)$. By 4.465 ("The logical product of a tautology and a proposition says the same as the proposition"), we have that $[Pb \vee \neg Pb] \wedge b = a$ says the same as $b = a$.[21] Hence $\exists x\,([Px \vee \neg Px] \wedge x = a)$ says the same as $\exists x\, x = a$. Consequently, this latter formula, a *Scheinsatz* according to 5.534, says the same as $Pa \vee \neg Pa$, which is therefore its appropriate translation. Given that Tractarian commitments force one of Wittgenstein's examples of *Scheinsätze* to be translated into a proposition expressed in correct logical notation, it is reasonable to infer that *Scheinsätze*, in so far as their *Schein* status is due to their containing the equality sign (cf. 6.2), generally are to be translated into propositions expressed in correct notation. Rather than being a fault of our translation procedures, our ability to translate *Scheinsätze* into Wittgensteinian notation is therefore a confirmation of their conformity with Tractarian doctrine.

2.4. **Ramsey's Reading.** Just a few years after the publication of the *Tractatus*, Ramsey made what was likely the first attempt to formulate Wittgenstein's identity convention precisely. His work was left unfinished, but eventually published posthumously as (Ramsey 1991). The interpretation Ramsey offers is the weakly exclusive reading. His formulation contains two clauses. First, "two different constants must not have the same meaning" (p. 159), which is just what we have called the Satisfaction Principle. Second, "an apparent variable cannot ((have)) the value of any letter occurring in its *scope*, unless the letter is a variable *apparent in that scope*" (ibid., emphases and editorial parentheses in the original). In other words, a bound variable must be read exclusively from all variables occurring free in its scope, which is precisely the inward-looking procedure of the weakly exclusive reading. Ramsey thus provided the same characterization of Wittgenstein's identity convention that Hintikka arrived at independently over 30 years later.

While Ramsey does not explicitly address either of the strongly exclusive readings, his comments regarding particular propositions rule out both of them. He remarks, for example, that in the formula $fa \,.\!\vee\!.\, (x). \sim fx$, a is not to be excluded from the range of the variable x (p. 158). This rules out the broad strongly exclusive reading, which would take the scope of a to be the entire formula. However, Ramsey says that we can "exclude a from the range of 'x' [...] by making 'a' occur in its scope", as happens in the formula $(\exists x).F(x, a)$ (p. 160). This example thus rules out the narrow strongly exclusive interpretation, which would take a to have no scope at all. Furthermore, Ramsey rejects the outward-looking procedure common to both of the strongly exclusive readings:

> It is clear that we must be able to treat '$(x).fx$' as a unit having a fixed meaning independent of what else occurs in the proposition. (p. 158)

This condition doesn't hold in either of the strongly exclusive readings, which dictate that the meaning of a quantified formula depends on the context in which it occurs. For example, according to these readings, the range of x in $\forall x\, fx$ is unrestricted, but in the formula $\exists y(\forall x\, fx \supset fy)$, its value cannot coincide with that of y.

Ramsey was in an ideal position accurately to characterize Wittgenstein's intended convention for identity, for he had the benefit of extensive consultation with Wittgenstein himself. In the summer of 1923, Ramsey visited Wittgenstein in Austria for two weeks, during which time the two discussed the *Tractatus* in detail for five hours per day.[22] They continued to correspond after Ramsey returned to Cambridge; later that year Ramsey reported in a letter to Wittgenstein that he was having "difficulty in expressing without = what Russell expresses by $(\exists x): fx \,.\, x \neq a$" (McGuinness 2008, p. 144). Wittgenstein's response has not been preserved, but we do know that he provided a translation of this Russellian proposition into his own notation, for Ramsey later replied, "Thanks for giving me the expression $fa \,.\!\supset\!.\, (\exists x, y) \,.\, fx \,.\, fy : \sim fa \supset (\exists x) fx$." (ibid., p. 146).[23] The exact date of composition of Ramsey's paper is unknown, but it was most likely written soon after the date of this correspondence (December 27, 1923), because Ramsey reproduces Wittgenstein's formula in his paper.

As noted earlier, none of the examples from the text of the *Tractatus* can help determine whether Wittgenstein's intended convention for identity was the weakly exclusive reading or the narrow strongly exclusive reading. However, since Ramsey's paper was written shortly after consulting with Wittgenstein about his identity convention, and Ramsey both explicitly formulates the weakly exclusive interpretation and rejects the strongly exclusive readings, it is plausible that Wittgenstein further clarified the nature of his convention in his letter to Ramsey in such a way that ruled out the narrow strongly exclusive reading as well. Ramsey's extensive consultation with Wittgenstein gives his characterizations of the convention prima facie legitimacy; in light of this evidence we believe that Wittgenstein's intended convention for removing the identity sign was the weakly exclusive reading. We would also consider it uncharitable to attribute either of the strongly exclusive readings to Wittgenstein, which have certain disadvantages vis-à-vis the weakly exclusive reading. First, the strongly exclusive readings have the added complication of requiring us to keep track of the set of free variables V whose values are excluded from the ranges of subsequent bound variables. Second, the meaning of an expression under either strongly exclusive reading is not fixed solely by its component expressions, but is also affected by the occurrence of variables outside of its scope, as Ramsey notes. Thus we attribute to Wittgenstein the weakly exclusive interpretation.

2.5. **Exclusive Interpretations in the Literature.** It is necessary at this point to address a terminological confusion that has arisen in the literature. Hintikka (1956) originated the terminology of weakly and strongly exclusive interpretations. According to his original formulation, given a formula $\exists x \phi$ written in Wittgenstein's notation, the two interpretations differ on what variables must be read exclusively from x. Under the weakly exclusive interpretation, "the value of x must not concur with ... the variables which occur freely within the range of $[\exists x]$", while under the strongly exclusive interpretation, they must not concur with "the variables within the range of which $[\exists x]$ occurs" (p. 230).[24] The reader will note that these interpretations precisely coincide, respectively, with our "inward-looking" and "outward-looking" procedures for restricting the range of a bound variable, and thus that our terminology of weakly and strongly exclusive interpretations matches Hintikka's own. For both of Hintikka's readings, the question of whether two bound variables must be read exclusively from each other arises *only* if their scopes overlap. Bound variables with non-overlapping scopes, e.g. x and y in $\exists x\, Fx \wedge \exists y\, Fy$, are not restricted from having concurring values under either of Hintikka's exclusive interpretations.

Hintikka's interpretations are discussed in Floyd (2002); McGray (2006) follows Floyd's treatment. The terminology of weakly and strongly exclusive interpretations in these papers, however, fails to follow Hintikka's original formulations:

> The weakly exclusive reading applies only to variables within the scope of a sequence of quantifiers containing the variables. The values of such variables must be distinct. (McGray 2006, p. 158)[25]

By "sequence of quantifiers," McGray cannot mean just immediately consecutive occurrences of quantifiers, as in a prenex string, for he interprets the formula $\forall x(\exists yLxy \supset \exists zAzx)$ as meaning the same as the FOL$^{=}$-formula $\forall x(\exists y(x \neq y \wedge Lxy) \supset \exists z(x \neq z \wedge Azx))$ (ibid., p. 157). But Hintikka's weakly exclusive interpretation does not in fact require any two bound variables with overlapping scopes to be read exclusively from one another. As was shown earlier, in the formula $\forall y(\forall xPx \supset Py)$, no variables occur free in the scopes of either y or x, so under the weakly exclusive interpretation, neither of their ranges are restricted. They *are* restricted, however, under Hintikka's strongly exclusive reading. Thus, what Floyd and McGray refer to as Hintikka's weakly exclusive interpretation is in fact a strongly exclusive interpretation.

What Floyd and McGray refer to as Hintikka's strongly exclusive interpretation turns out to be very different from anything Hintikka ever intended:

> The strongly exclusive reading requires each distinct bound variable throughout the entire proposition to have a range restricted by every previously occurring quantifier. Bound variables are dependent upon all other bound variables in the entire proposition. (ibid.)[26]

According to this characterization, the values of any two bound variables in the same proposition must not concur, even if their scopes do not overlap. As shown above, however, both of Hintikka's interpretations are only applicable to variables with overlapping scopes. In essence, Floyd and McGray characterize the difference between the weakly and strongly exclusive interpretation as hinging on whether all variables in a given proposition are to be read exclusively, or only all those with overlapping scopes. In fact, Hintikka's actual interpretations differ according to whether all variables with overlapping scopes should be read exclusively, or only some of them should.

To prevent further confusion, we refer to the requirement that all bound variables in a given proposition be read exclusively from each other as the *roughshod* interpretation. As it stands the interpretation is not completely clear, for it does not provide direction for handling propositions in which the same variable letter is bound by more than one quantifier (like x in both propositions of 5.5321). Floyd provides two possible clarifications of the roughshod interpretation.

According to the first, "the same variable letter ... bound by different quantifiers in different parts of one sentence" should be "in effect read ... as a different letter in its two differing bound occurrences" (Floyd 2002, p. 325). The third proposition of 5.532, $(\exists x, y).f(x, y). \vee .(\exists x).f(x, x)$, constitutes a counterexample to this reading. It would have the same truth conditions as $\exists x \exists y(f(x, y) \vee \exists z\, f(z, z))$ under the strong readings, which are distinct from those of the Russellian formula $(\exists x, y).f(x, y)$, for the latter can be satisfied in a one-element domain, while the former cannot.

Floyd's alternative clarification of the roughshod interpretation is to "require that the same letter, even if bound in different occurrences by distinct quantifiers, be instantiated by names with the same Bedeutung throughout the sentence as a whole" (ibid.). This requirement is not sufficiently precise to fix the semantics of a formula such as $\forall x\, Fx \wedge \exists x\, Gx$, in which the same variable letter is bound by multiple kinds of quantifiers. Let us assume that quantifier precedence decreases from left to right. Then Floyd's second clarification would presumably require reading both $\forall x\, Fx \wedge \exists x\, Gx$ and $\forall x\, Fx \wedge \forall x\, Gx$ as $\forall x(Fx \wedge Gx)$ (understood classically), obliterating the difference between the existential and the universal quantifier in the second conjunct, as well as the standard mechanism of scope demarcation. Given the systematic awkwardness of this interpretation, as well as the complete lack of textual support, this second version of the roughshod interpretation also could not have been Wittgenstein's intended convention.[27]

As should now be clear, a formula *qua* syntactic expression will in general have different truth conditions when interpreted according to any of the exclusive conventions than when it is interpreted in the standard inclusive manner. For example, $\exists x \exists y(Fx \wedge Fy)$ is true under all of the exclusive interpretations only if at least two objects have F, but under the inclusive interpretation it is also true if F holds of just one object. Since the truth conditions of formulas differ between exclusive and inclusive interpretations of the variables, the valid schemas and inference rules will obviously also differ between these interpretations. Some commentators have missed this point, for their criticisms of Wittgenstein's notation are based on the erroneous assumption that all schemas and inferences valid under the inclusive interpretation must continue to hold literally when variables are interpreted exclusively.

Bogen (1981, p. 74) argues that the proposition "at least two things have the property F" cannot, as Wittgenstein implies at 5.532, be expressed in an identity-free notation as $(\exists x)(\exists y)(Fx \,\&\, Fy)$, because this formula follows from a proposition that is true when only one thing has F:

> [...] [This formula] is true if there is at least one name whose substitution for the variable in the function "Fx" produces a true proposition, and one name whose substitution for the variable in "Fy" produces a true proposition. This condition is met by a world in which two or more different things have the property, F, but it is met just as well by a world containing only one object with the property. Suppose the name of the only F-er is "a." Substituting "a" in first for "x" and then for "y," we get the true proposition "$Fa \,\&\, Fa$" even though the proposition "at least two things have property F" is false. (ibid.)

In this passage the following rule of existential generalization is taken for granted: from any formula $\phi(a)$ we may infer $\exists x\phi(x)$, even if a is not fully indicated in $\phi(a)$, that is, even if a still occurs in $\exists x\phi(x)$.[28] This rule is of course valid when variables are interpreted inclusively, but under all of the exclusive interpretations considered here it is invalid. In the *Notebooks*, Wittgenstein makes just this point:

> I believe that it would be possible wholly to exclude the sign of identity from our notation and always to indicate identity merely by the identity of the signs (and conversely). In that case, of course $\phi(a, a)$ would not be a special case of $(x, y).\, \phi(x, y)$, and $\phi(a)$ would not be a special case of $(\exists x, y).\, \phi x \,.\, \phi y$. (1961a, p. 34)

Thus $(\exists x, y).\, \phi x \,.\, \phi y$ cannot be inferred from ϕa, so the version of existential generalization stated above is invalid in Wittgenstein's notation. To find concrete counterexamples, one need only consider the two particular instances of existential generalization that Bogen assumes must hold in a Tractarian notation:

(1) $(Fa \wedge Fa)$ implies $\exists y(Fa \wedge Fy)$, and
(2) $\exists y(Fa \wedge Fy)$ implies $\exists x\exists y(Fx \wedge Fy)$.

Under the weakly exclusive and broad strongly exclusive readings, the first inference is invalid, because $(Fa \wedge Fa)$ is true if and only if a has F, while $\exists y(Fa \wedge Fy)$ is true if and only if both a and some other object have F. Under the narrow strongly exclusive reading, the second inference is invalid; $\exists y(Fa \wedge Fy)$ holds as long as a has F, while

$\exists x \exists y(Fx \wedge Fy)$ requires that F hold of at least two objects. Bogen's criticism is therefore based on inappropriately imposing requirements of an inclusive interpretation on Wittgenstein's exclusive interpretation of the variables.[29]

In a recent criticism, Landini (2007, p. 254) claims that Wittgenstein's proposal of "using scope alone to track exclusivity raises an important difficulty for formulating a deductive system for exclusive quantifiers". In a fashion similar to Bogen, Landini simply assumes that all schemas that are valid when variables are interpreted inclusively must remain valid under an exclusive interpretation. He considers the schema $(x)(B \supset Ax) \,.\supset.\, B \supset (x)Ax$ (where x does not occur free in B), whose instances are all axioms of classical quantification theory under the inclusive interpretation of the variables. This schema fails to be universally valid under each of the exclusive interpretations, as shown by the following instances:

(1) $(x)(Fa \supset Gx) \,.\supset.\, Fa \supset (x)Gx$
(2) $(x)((\exists y)Fy \supset Gx) \,.\supset.\, (\exists y)Fy \supset (x)Gx$

Under the weakly exclusive interpretation (1) is invalid, because x is read exclusively from a in the antecedent but not in the consequent. Likewise, (2) is invalid under both the narrow and broad strongly exclusive interpretations, which read y exclusively from x in the antecedent but not in the consequent.[30] "To rectify this problem," Landini chooses to "abandon Wittgenstein's idea that the scope of quantifiers determines exclusivity" (ibid., p. 255), and instead develops a deductive calculus using quantifiers superscripted with the variables whose values are excluded from the ranges of the quantifiers' bound variables.[31] Of course, the above situation is only a "problem" if one falsely assumes that all valid axiom schemas of classical quantification theory also hold when variables are interpreted exclusively.

Wittgenstein proposes a new logical notation that uses the same symbols as standard first-order logic (sans the identity sign), but *interprets* some of those symbols, in particular the variables, in a *different* manner. Bogen and Landini nevertheless criticize this language for failing to function like the standard language in which variables are inclusively interpreted. The groundlessness of this complaint is shown by consideration of the following analogous situation. If we construct a propositional logic in which the symbol $\wedge$ is used to mean "or", and $\vee$ is used to mean "and", then in this language $P \wedge \neg P$ is a tautology and $P \vee \neg P$ is not. It would be entirely baseless to complain that $P \vee \neg P$

ought to be a tautology. The expressive capacities of this alternate language are obviously identical to those of standard propositional logic. Analogously, one cannot expect all valid schemas of classical first-order logic to hold in Wittgenstein's alternative logic. We can however legitimately insist that if Wittgenstein's notation is to be considered an adequate alternative to standard first-order logic, the *translation* of every valid sentence in the Russellian notation must be valid in the Wittgensteinan notation. This is exactly what Lemmas 1 and 2 above ensure.

3. The N-Operator

For the duration of section 3 we set aside the convention for eliminating the identity sign and focus on Wittgenstein's claim that every proposition can be represented as the result of iterated applications of a single operator to elementary propositions:

> Every truth-function is a result of the successive application of the operation "(- - - - - T)($\xi, \ldots .$)" to elementary propositions.
> This operation denies all the propositions in the right-hand bracket and I call it the negation of these propositions. (5.5)

The description of this operation uses Wittgenstein's symbolism for finite truth functions, introduced at 4.442: (- - - - - T)($\xi, \ldots .$) is true only for the last row of a truth table listing each of the propositions in ($\xi, \ldots .$) as having independent truth values and arranged in the manner given at 4.31, i.e., the row representing the case in which all of the propositions in ($\xi, \ldots .$) are false. This is a generalization of (F F F T)(p, q)—the operation of binary joint denial known variously as NOR, the Sheffer stroke, or the dagger—such that ($\xi, \ldots .$) can be any finite list of enumerated propositions.

Wittgenstein then introduces an alternative notation:

> [...] I write instead of "(- - - - - T)($\xi, \ldots .$)", "N($\bar{\xi}$)".
> N($\bar{\xi}$) is the negation of all the values of the propositional variable ξ. (5.502)[32]

This constitutes a further generalization of the operation, for the N-operator is now applied not to a finite list but to a class, which may have an infinite number of elements. This class consists of all the values of a propositional variable.

By a *propositional variable*, Wittgenstein understands any variable whose values are propositions. A *variable proposition* is an open sentence, i.e. the result of replacing, in a proposition, a constituent by a variable. Finally, a *propositional function* is a function that sends names to propositions in such a way that a name a is mapped to the result $\phi(a)$ of replacing the variable x in a variable proposition $\phi(x)$ with a.

We begin by reviewing, in 3.1, Wittgenstein's methods of describing the values of a propositional variable and explain the conditions under which a proposition in N-notation can exhibit generality. As is well known, Fogelin has cast doubt on the expressive adequacy of the N-operator with respect to certain general propositions. In 3.2, we defend Geach's rebuttal of Fogelin's charge on the basis of our analysis in 3.1, and show in 3.3 how Geach's proposed notation is justified on Tractarian grounds. We end section 3 with a presentation of N-logic, a system that employs N as its sole logical operator.

3.1. **Propositional Variables and Generality.** Wittgenstein offers several ways in which the values of a propositional variable may be determined:

> An expression in brackets whose terms are propositions I indicate — if the order of the terms in the bracket is indifferent — by a sign of the form "$(\bar{\xi})$". "ξ" is a variable whose values are the terms of the expression in brackets, and the line over the variable indicates that it stands for all its values in the bracket.
> (Thus if ξ has the 3 values P, Q, R, then $(\bar{\xi}) = (P, Q, R)$.)
> The values of the variables must be determined.
> The determination is the description of the propositions which the variable stands for.
> How the description of the terms of the expression in brackets takes place is unessential.
> We may distinguish 3 kinds of description: 1. Direct enumeration. In this case we can place simply its constant values instead of the variable. 2. Giving a function fx, whose values for all values of x are the propositions to be described. 3. Giving a formal law, according to which those propositions are constructed. [...] (5.501)

Just as in the earlier case of translating between Russellian and identity-free notation, Wittgenstein does not explicitly state any syntactical rules for the application of the N-operator (presumably because he thinks this is "easy" (5.503)). In §3.4, we provide precise syntactical rules for N-logic and show how to translate between it and $\mathrm{FOL}^{=}$.

For the cases in which the values of the propositional variable ξ are specified in either the first or the second way of 5.501, Wittgenstein illustrates the result of applying the N-operator by means of equivalent Russellian formulas:

> If ξ has only one value, then $\mathrm{N}(\bar{\xi}) = {\sim}p$ (not p), if it has two values then $\mathrm{N}(\bar{\xi}) = {\sim}p\,.\,{\sim}q$ (neither p nor q). (5.51)

This shows that if N is applied to an enumerated list of propositions (i.e. a class described by the first method of 5.501), the result is equivalent to the joint negation of all the listed propositions. The N-operator thus functions just like the Sheffer stroke when it is applied to exactly two propositions. It is then clear from Sheffer's result that any proposition of the sentential calculus can be represented by means of the N-operator alone. However, it remains for Wittgenstein to explain how quantified propositions are to be represented in his notation. These are shown to be the results of applying N to classes of propositions described in the second way:

> If the values of ξ are the total values of a function fx for all values of x, then $\mathrm{N}(\bar{\xi}) = {\sim}(\exists x).fx$. (5.52)

By "function" Wittgenstein here just means a propositional function. A propositional function can serve as a means for determining the values of a propositional variable, because every propositional function corresponds to a class of propositions:

> [...] corresponding to any propositional function $\phi\hat{x}$, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx. (*Principia*, pp. 15-16)

Wittgenstein clearly appeals to this doctrine in the *Tractatus*:

> If we change a constituent part of a proposition into a variable, there is a class of propositions which are all the values of the resulting variable proposition. [...] (3.315)

This is a procedure for describing the values of a propositional variable according to the second method of 5.501. We replace a fixed expression in a proposition with a variable, thereby obtaining a variable proposition, which in turn serves to specify a propositional function. We then simply let the values of the propositional variable be all the values of this propositional function. The propositional variable ξ employed at

5.52, for example, is the result of taking a proposition, say, fa, replacing the name a with the variable x to produce a variable proposition, and then letting all of the values of the corresponding function be values of ξ.

Only by applying N to a class of propositions that is described by means of a propositional function do we arrive at propositions that exhibit generality:

> That which is peculiar to the "symbolism of generality" is firstly, that it refers to a logical prototype, and secondly, that it makes constants prominent. (5.522)

The propositional function fx indeed makes the constant expression f prominent, for this mark is shared by all the values of the function. It is also a prototype, for fx is not itself a proposition, but can be transformed into one if x is replaced by a constant name.[33] But the members of an enumerated list of propositions described by the first method of 5.501 have neither of these properties. They are not prototypes but rather fully determinate propositions, and they do not bring to prominence particular constants, for there may very well be no common marks shared by the members of a particular enumerated list of propositions—one need only consider (fa, gb). Wittgenstein thus claims that the application of N to the class of propositions in 5.52 corresponding to the values of the propositional function fx results in a general proposition, while applications to the enumerated lists of propositions in 5.51 do not. This does not, as one might think, violate Wittgenstein's claim that "how the description of the terms of the expression in brackets takes place is unessential" (5.501). The method of description may be essential for determining whether a proposition exhibits generality, but it is of no consequence to the resulting proposition's truth value. Suppose given two methods of description, A and B. Then applying N to the class of propositions determined by method A results in the same truth function of elementary propositions as applying N to the class of propositions determined by method B, provided that methods A and B describe the same class of propositions. In other words, N is an extensional operator on classes.

Having demonstrated how the N-operator functions in various cases, Wittgenstein finally claims that *every* proposition can be represented as the result of iterated applications of N, provided that the base set of elementary propositions is given (6.001).[34] Thus, mirroring his proposal for a notation that eliminates the sign of identity through the exclusive interpretation of the variables, Wittgenstein proposes that the quantifiers and sentential operators of *Principia* be eliminated through the use of the N-operator. The resulting N-logic, he argues, has all of the expressive capacities of the Russellian system. We should note that the *Tractatus* supplies no examples of applying N to propositions described by the third method of 5.501 (giving a formal law for the construction of propositions). This does not concern us here, since the first two methods alone already supply N-logic with the resources for an expressively complete first-order logic, which is the extent of our interest in this essay.

3.2. **The Fogelin-Geach Debate.** Wittgenstein's claim concerning the expressive power of the N-operator is disputed by Fogelin (1976), who argues that mixed multiply general propositions (e.g. $\forall x \exists y\, Fxy$) cannot be represented using the N-operator, and thus that Wittgenstein's system is expressively incomplete. In response to Fogelin's charge, Geach (1981) introduces "an explicit notation for a class of propositions in which one constituent varies" (p. 169), and argues that the logic of the *Tractatus* becomes expressively complete when supplemented by this notation. Fogelin remains unconvinced in his (1987), claiming that Geach's notation violates Tractarian doctrine.[35] In the following we defend Geach's notation on the ground that it serves to indicate propositional functions, and thus is licensed by 3.315 and 5.501. We further argue that Fogelin's reading of Tractarian doctrine is mistaken.

Fogelin argues that homogeneous multiply general propositions are easy to render using the N-operator:

> [...] let ξ have as its values the values of the function fxy for all values of x and y, i.e., faa, fab, fba, fac, etc. Since $\mathrm{N}(fxy)$ gives the joint denial of all those propositions that are the values of the propositional function fxy, it is evident we have produced a proposition equivalent to [...] "$(x)(y) -fxy$". We can next bring this resulting proposition under the operator N, i.e., just deny it, and this gives us a result equivalent to [...] "$(\exists x)(\exists y) fxy$." (1987, pp. 78-79)

He claims, however, that the N-operator cannot produce mixed quantification, due to the way that it handles expressions with multiple variables:

> When we apply the operator N to the propositions that are the values of the function fxy, both argument places under the function are handled at once in the same way,

> i.e., both variables are captured. So whatever kind of quantifier emerges governing one of the variables, that same kind of quantifier must emerge governing the other. (ibid., p. 79)

It is thus impossible to existentially quantify over one variable and universally over another using the N-operator, according to Fogelin's construal, for N effectively binds each free variable occurring in the expression in brackets simultaneously.

Problematic assumptions underlying Fogelin's critique are brought to the fore in the following demonstration. Let us attempt to construct the mixed multiply general proposition $\forall x \exists y\, fxy$ using the N-operator. We first use the propositional function fay to describe a class of elementary propositions. We then apply N to this class to get $\mathrm{N}(fay)$, which is equivalent to $\sim\exists y\, fay$. Since $\forall x \exists y\, fxy$ is equivalent to $\sim\exists x \sim\exists y\, fxy$, it seems clear that our next step must be to apply N to all propositions having a form equivalent to $\sim\exists y\, fxy$, which we should be able to describe with a propositional function. We therefore replace a in the proposition $\mathrm{N}(fay)$ with the variable x to produce $\mathrm{N}(fxy)$, and finally apply N to this class to arrive at $\mathrm{N}(\mathrm{N}(fxy))$. Yet $\mathrm{N}(\mathrm{N}(fxy))$ is precisely the expression that Fogelin claims is equivalent to $\exists x \exists y\, fxy$. Clearly something in this construction has gone wrong.

It is not difficult to see where the problem lies. After constructing the proposition $\mathrm{N}(fay)$, we attempted to produce a propositional function by replacing a with the variable x. However, the resulting expression $\mathrm{N}(fxy)$ fails, according to Fogelin's construal, to be a propositional function; it is instead what he argues is the proper representation of $\forall x \forall y \sim fxy$ in Tractarian notation. Thus in our attempt to produce a propositional function that would correspond to an entire class of propositions, we instead wound up with a single determinate proposition. Using Fogelin's notation, we are unable to distinguish between the case in which $\mathrm{N}(fay)$ is turned into a propositional function by replacing a with x, and the case in which we apply N to the class of values of the propositional function fxy, for both of these must be symbolized as $\mathrm{N}(fxy)$.

This is a systematic problem under Fogelin's treatment of Tractarian logic. Wittgenstein asserts unconditionally that "if we change a constituent part of a proposition into a variable, there is a class of propositions which are all the values of the resulting variable proposition" (3.315). So we should *always* be able to indicate a propositional function by letting part of a proposition vary, and thereby describe a class of propositions according to the second method of 5.501. On Fogelin's reading, this technique succeeds in producing elementary propositional functions only (i.e. propositional functions whose values are elementary propositions). But the technique fails to produce propositional functions whose values are complex propositions. In a logical notation using only the N-operator, every complex proposition has the form N(…). Fogelin's claim that $\mathrm{N}(fxy)$ is the correct representation of $\forall x \forall y \sim fxy$ shows that he takes N to bind *all* free variables occurring in its scope. Taking part of a complex proposition and making it variable results in an expression that is *still* of the form N(…). Thus, the new variable introduced by the 3.315 procedure is automatically bound by the N-operator, resulting in a determinate proposition rather than an open sentence. This shows that Fogelin's construal of the N-operator provides no way of representing complex propositional functions. As a result, he has no means of using the second method of 5.501 to describe classes of complex propositions. Fogelin therefore cannot represent *any* quantified complex propositions using the N-operator, because the second method is necessary for generality. Since 3.315 declares that any proposition can be turned into a propositional function, Fogelin's exclusion of all complex propositional functions cannot be faithful to the text.

Fogelin concludes that $(\forall x)(\exists y) fxy$ cannot be represented with the N-operator, "given the explicitly stated notation procedures of the Tractatus" (1987, p. 79). We must however point out that the *Tractatus* in fact provides *no* explicit notation procedures like those that Fogelin uses. Recall that 5.52 only says that *if* the propositional variable ξ has as its values all the values of the propositional function fx, then applying N to $\bar{\xi}$ results in a proposition that is equivalent to $\sim(\exists x).\, fx$. It does *not* say that the symbolization of this proposition must be $\mathrm{N}(fx)$, as Fogelin assumes.[36] Wittgenstein's remarks constitute merely a proposal for a notation using only the N-operator with guidelines that such a notation should follow, not an actual set of syntactical rules. An interpretation of Wittgenstein's N-operator that is faithful to the text must respect the requirement at 3.315 that any proposition may be turned into a propositional function.

This requirement is fulfilled with the use of a piece of notation introduced by Geach:

> We need (something [Wittgenstein] does not himself provide) an explicit notation for a class of propositions in which one constituent varies. I shall write '$\mathrm{N}(x\colon fx)$'[37] to mean joint denial of the class of propositions got by

> substituting actual names for the variable in the propositional function (represented by) 'fx'. (1981, p. 169)

Geach thus provides a notation that clearly distinguishes between free and bound variables, and hence between propositional functions and determinate propositions, thereby fully satisfying the requirements of 3.315. We can now transform any proposition $\varphi(a)$ into a propositional function $x\colon \varphi(x)$ by replacing one or more instances of some name a in φ with a fresh variable x and prefixing the expression with the variable x and a colon; by enclosing this expression within parentheses we denote the class of all the propositional function's values. With Geach's notation in hand, we can successfully construct a proposition equivalent to $(\forall x)(\exists y)fxy$ using the N-operator. We begin with the elementary proposition fab, and then make b variable, arriving at the propositional function $y\colon fay$. Applying N to the class of its values, $(y\colon fay)$, produces $\mathrm{N}(y\colon fay)$, which is equivalent to $\sim(\exists y)fay$. We then replace a with a variable in order to construct the propositional function $x\colon \mathrm{N}(y\colon fxy)$. Application of N to the class of propositions $(x\colon \mathrm{N}(y\colon fxy))$ yields $\mathrm{N}(x\colon \mathrm{N}(y\colon fxy))$, which is equivalent to $(\forall x)(\exists y)fxy$.

The introduction of a device like Geach's into a language using the N-operator is therefore faithful to, and indeed required to satisfy, Wittgenstein's intentions in the *Tractatus*, for it allows 3.315 to be fully realized. Fogelin nevertheless accuses Geach of violating Tractarian doctrine with his notation.[38] As Fogelin understands Geach's symbolism, if the domain of objects is infinite, then the expression $(x\colon \mathrm{N}(fx))$ is merely shorthand for an infinitely long enumerated list of propositions each of which is the result of an application of N (1987, p. 80). He thus construes Geach's expression $\mathrm{N}(x\colon \mathrm{N}(fx))$ (which is equivalent to $(\forall x)fx$) as shorthand for $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb), \mathrm{N}(fc), \ldots)$ (1982, p. 126).[39] Fogelin argues that this construction is utterly refuted by 5.32:

> All truth-functions are results of the successive application of a finite number of truth-operations to elementary propositions.

Fogelin takes this to be an assertion that the construction of any proposition must not involve an infinite number of applications of N. According to him,

> [...] Wittgenstein never says that the construction of a proposition may involve infinitely many applications of the operation N. In fact, he says just the opposite [...]. (Fogelin 1976, p. 73)

This is simply a misreading of the text, for 5.32 does *not* say that all truth-functions are results of a finite number of *applications* of truth-operations, as Fogelin claims, but rather, merely a finite number of *truth-operations*.[40] Indeed, Wittgenstein argues that they are the result of just one operation, namely N.[41] In defense of Geach, many interpreters have tried to argue that constructions using Geach's notation do not involve an infinite number of applications of N. With respect to Fogelin's charge, such an argument is moot; Wittgenstein simply does not claim at 5.32 that every proposition is the result of a finite number of applications of N, so Geach's notation stands in no need of defense from this accusation.

3.3. **Circumflex and Horizontal Bar.** A more detailed elaboration of the N-notation by Wittgenstein himself might well have resulted in a device equivalent to Geach's. While Fogelin and Geach agree that the *Tractatus* lacks such a device, in some sense it is already present in the *Tractatus* when considered within its immediate contemporaneous context, specifically *Principia*.

Our point is this: The second way of describing the base for an application of N consists in the specification of "a [propositional] function fx, whose values for all values of x are the propositions to be described" (5.501). Now strictly speaking, Wittgenstein should have spoken not of a function fx, but rather of a function $f\hat{x}$, for reasons laid out by Whitehead and Russell in *Principia* (p. 15), with which Wittgenstein was, of course, intimately familiar:

> [...] if "x is hurt" and "y is hurt" occur *in the same context*, [...] then according to the determinations given to x and y, they can be settled to be (possibly) the same proposition or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function. When we wish to speak of the propositional function corresponding to "x is hurt," we shall write "$\hat{x}$ is hurt." Thus "$\hat{x}$ is hurt" is the propositional function and "x is hurt" is an ambiguous value of that function. Accordingly though "x is hurt" and "y is hurt" *occurring in the same context* can be distinguished, "$\hat{x}$ is hurt" and "$\hat{y}$ is hurt" convey no distinction of meaning at all. More generally, ϕx is an ambiguous value of the

> propositional function $\phi\hat{x}$, and when a definite signification a is substituted for x, ϕa is an unambiguous value of $\phi\hat{x}$.

In the context of 5.501, this distinction is not critical, first because Wittgenstein explicitly speaks of the *function* fx, thus alerting us to the, as it were, bound status of x, and secondly because there is no possibility for confusion with only one individual variable occurring in fx.[42] But in the kind of case considered by Fogelin, the distinction is crucial, for clearly $f\hat{x}y$, $fx\hat{y}$, and $f\hat{x}\hat{y}$ are, in general, very different functions. As Wittgenstein was well aware (cf. 4.0411), the circumflex alone is insufficient to effect the scope distinctions necessary for the expression of nested quantifications; however, in conjunction with the horizontal bar that is placed on top of the propositional variable ξ, the circumflex constitutes an equivalent of Geach's notation: Instead of $\mathrm{N}(x : \phi(x))$ we can write $\mathrm{N}\,\overline{\phi(\hat{x})}$, the circumflex indicating the bound variable and the horizontal bar the extent of its scope.[43] In nested applications, the scope of a bound variable will be indicated by the lowest horizontal bar above that variable's circumflex. Here are some examples of homogeneous and heterogeneous multiple quantifications in standard and circumflex notation:

> $\forall x \exists y\, fxy$ corresponds to $\mathrm{N}\,\overline{\mathrm{N}\,\overline{f\hat{x}\hat{y}}}$
>
> $\exists x \forall y\, fxy$ corresponds to $\mathrm{N}\,\mathrm{N}\,\overline{\mathrm{N}\,\overline{\mathrm{N}\, f\hat{x}\hat{y}}}$
>
> $\exists x \exists y\, fxy$ corresponds to $\mathrm{N}\,\mathrm{N}\,\overline{\mathrm{N}\,\mathrm{N}\,\overline{f\hat{x}\hat{y}}}$
>
> $\forall x \forall y\, fxy$ corresponds to $\mathrm{N}\,\overline{\mathrm{N}\,\mathrm{N}\,\overline{\mathrm{N}\, f\hat{x}\hat{y}}}$

Ascribing the plan for such a notation to Wittgenstein is not only plausible, given the notational resources with which he was familiar[44]; it has the additional advantage of making sense of the horizontal bar, which otherwise serves no discernible purpose. Indeed, Fogelin completely ignores it, and not even Geach provides a reason for the bar's presence in Wittgenstein's symbolism.

Ramsey's work again lends support to our view. In his review of the *Tractatus* (Ramsey 1923) we find the following passage:

> [Wittgenstein's notation "$(T\,T\text{-}T)(p, q)$"] does not in any way require p, q to be elementary propositions; and it can be extended to include propositions containing apparent variables. Thus p, q may be given not by enumeration but as all values of a propositional function,

i.e., all propositions containing a certain expression (defined as "any part of a proposition which characterizes its sense" (3.31)); and $(\text{- - - - - -}T)(\bar{\xi})$, where the solitary T expresses agreement only with the possibility that all the arguments are false, and $\bar{\xi}$ is the set of values of $f\hat{x}$, is what is written ordinarily as $\sim\,:(\exists x).fx$. (ibid., 470-71)

In Ramsey's passage, we find in one place both of the ingredients for the notation we have suggested Wittgenstein envisaged: the circumflex to indicate the independent variable of the propositional function with whose help the values of the propositional variable ξ are to be specified, and the horizontal bar that transforms the expression of a propositional function into one for the class of all its values. The presence of the circumflex, in particular, is remarkable, as it does not occur in the *Tractatus* itself. Although the review was written before Ramsey's consultations with Wittgenstein in the summer of 1923, the passage demonstrates how natural it was for someone working in the *Principia* tradition to assume the availability of circumflex notation.

3.4. N-**Logic.** We can now precisely describe a logic based on Wittgenstein's N-operator; for ease of typesetting, we revert to Geach's notation for propositional functions. The language of N-logic is $\mathcal{L}_{\mathrm{N}}$, which has the following primitive symbols:

(1) an unlimited stock of individual variables x, y, $x_0, x_1, \ldots$
(2) for each $n \in \omega$, an unlimited stock of n-ary predicate symbols $P, R, S, P_0, P_1, \ldots$, among which the binary equality sign $=$
(3) the symbol N
(4) left and right parentheses, the comma, and the colon as punctuation marks.

The $\mathcal{L}_{\mathrm{N}}$-formulas are defined inductively:

(1) If $x_1, \ldots, x_n$ are individual variables (not necessarily distinct), and R is an n-ary predicate symbol, then $Rx_1 \ldots x_n$ is an $\mathcal{L}_{\mathrm{N}}$-formula. (When R is $=$, we write $x = y$ rather than $= xy$.)
(2) If $\phi_0, \ldots, \phi_n$ are $\mathcal{L}_{\mathrm{N}}$-formulas, then so is $\mathrm{N}(\phi_0, \ldots, \phi_n)$.
(3) If ϕ is an $\mathcal{L}_{\mathrm{N}}$-formula and x is an individual variable, then $\mathrm{N}(x\!:\!\phi)$ is an $\mathcal{L}_{\mathrm{N}}$-formula.

We now define the notion of an assignment's satisfying an $\mathcal{L}_{\mathrm{N}}$-formula ϕ in a structure $\mathcal{U}$ by recursion on ϕ, in symbols: $\mathcal{U} \models \phi[\sigma]$.

(1) If R is an n-ary predicate symbol other than the equality sign, we have $\mathcal{U} \models Rx_1 \ldots x_n[\sigma]$ if and only if $R^{\mathcal{U}}$ holds of $(\sigma(x_1), \ldots, \sigma(x_n))$. $\mathcal{U} \models x = y[\sigma]$ if and only if $\sigma(x) = \sigma(y)$.

(2) $\mathcal{U} \models \mathrm{N}(\phi_0, \ldots, \phi_n)[\sigma]$ if and only if $\mathcal{U} \not\models \phi_0[\sigma], \ldots, \mathcal{U} \not\models \phi_n[\sigma]$.
(3) $\mathcal{U} \models \mathrm{N}(x\colon \phi)[\sigma]$ if and only if for all x-variants[45] τ of σ, $\mathcal{U} \not\models \phi[\tau]$.

Each of the variables $x_1, \ldots, x_n$ occurs free in $Rx_1 \ldots x_n$. The variables occurring free in $\mathrm{N}(\phi_1, \ldots, \phi_n)$ are precisely those occurring free in any of the ϕ_i, and the variables occurring free in $\mathrm{N}(x\colon \phi)$ are precisely those free in ϕ other than x. A *sentence* is a formula without any free occurrences of variables. For sentences ϕ of $\mathcal{L}_\mathrm{N}$, we have $\mathcal{U} \models \phi[\sigma]$ for some σ if and only if $\mathcal{U} \models \phi[\sigma]$ for all σ, because ϕ contains no free variables, and hence we may define ϕ's being true (*simpliciter*) in $\mathcal{U}$, $\mathcal{U} \models \phi$, as $\mathcal{U} \models \phi[\sigma]$ for all (equivalently, some) $\mathcal{U}$-assignments σ.

We can translate $\mathcal{L}_\mathrm{N}$ into $\mathrm{FOL}^=$ via the function $\mathcal{F}$, defined recursively as follows:

(1) $\mathcal{F}(Rx_1 \ldots x_n) = Rx_1 \ldots x_n$
(2) $\mathcal{F}(\mathrm{N}(\phi_0, \ldots, \phi_n)) = \neg\mathcal{F}(\phi_0) \wedge \ldots \wedge \neg\mathcal{F}(\phi_n)$
(3) $\mathcal{F}(\mathrm{N}(x\colon \phi)) = \forall x \neg\mathcal{F}(\phi)$

It is easy to see that for all $\mathcal{L}_\mathrm{N}$-formulas ϕ, all models $\mathcal{U}$, and all $\mathcal{U}$-assignments σ, we have

$$\mathcal{U} \models \phi[\sigma] \Leftrightarrow \mathcal{U} \models \mathcal{F}(\phi)[\sigma].$$

We can also translate in the reverse direction by means of the following recursively defined function $\mathcal{D}$ (we assume for the sake of simplicity that only $=$, $\neg$, $\vee$, and $\forall$ are primitive logical symbols of $\mathrm{FOL}^=$):

(1) $\mathcal{D}(Rx_1 \ldots x_n)$ is $Rx_1 \ldots x_n$; $\mathcal{D}(x = y)$ is $x = y$
(2) $\mathcal{D}(\neg\phi) = \mathrm{N}(\mathcal{D}(\phi))$
(3) $\mathcal{D}(\phi \vee \psi) = \mathrm{N}(\mathrm{N}(\mathcal{D}(\phi), \mathcal{D}(\psi)))$
(4) $\mathcal{D}(\forall x\phi) = \mathrm{N}(x\colon \mathrm{N}(\mathcal{D}(\phi)))$

Again, it is easy to see that for all $\mathrm{FOL}^=$-formulas ϕ, all models $\mathcal{U}$, and all $\mathcal{U}$-assignments σ, we have

$$\mathcal{U} \models \phi[\sigma] \Leftrightarrow \mathcal{U} \models \mathcal{D}(\phi)[\sigma].$$

Even though 5.32 does not, pace Fogelin, *require* a finite number of applications of N, the above function $\mathcal{D}$ nevertheless shows that every proposition of $\mathrm{FOL}^=$ can be constructed using a finite number of applications of the N-operator. Since clauses 2-4 replace each standard first-order logical operator with a finite number of occurrences of N, and every sentence of $\mathrm{FOL}^=$ contains a finite number of operators, the translation $\mathcal{D}(\phi)$ of any first-order formula ϕ will be the result of a finite number of successive applications of the N-operator.

4. First-Order Tractarian Logic

We now present a reconstruction of the first-order part $\mathcal{T}$ of Tractarian logic, based on the considerations of the preceding sections. $\mathcal{T}$ is, in effect, a combination of the weakly exclusive interpretation of the variables and N-logic as developed in the preceding sections. The language $\mathcal{L}_\mathcal{T}$ of $\mathcal{T}$ is just the language $\mathcal{L}_\mathrm{N}$, introduced in §3.4, but without the equality sign.

We now define the notion of an assignment's satisfying an $\mathcal{L}_\mathcal{T}$-formula ϕ in a structure $\mathcal{U}$ by recursion on ϕ, in symbols: $\mathcal{U} \Vdash \phi[\sigma]$.

(1) For atomic formulas $Rx_1 \dots x_n$, we have $\mathcal{U} \Vdash Rx_1 \dots x_n[\sigma]$ if and only if $R^\mathcal{U}$ holds of $(\sigma(x_1), \dots, \sigma(x_n))$.
(2) $\mathcal{U} \Vdash \mathrm{N}(\phi_0, \dots, \phi_n)[\sigma]$ if and only if $\mathcal{U} \nVdash \phi_0[\sigma], \dots, \mathcal{U} \nVdash \phi_n[\sigma]$.
(3) $\mathcal{U} \Vdash \mathrm{N}(x\colon\phi)[\sigma]$ if and only if for all x-variants τ of σ such that $\tau(x) \notin \{\sigma(y) \mid y \text{ occurs free in } \mathrm{N}(x\colon\phi)\}$, $\mathcal{U} \nVdash \phi[\tau]$.[46]

For *sentences* ϕ, we have $\mathcal{U} \Vdash \phi[\sigma]$ for some σ if and only if $\mathcal{U} \Vdash \phi[\sigma]$ for all σ, and hence we may define a sentence ϕ's being true in $\mathcal{U}$, $\mathcal{U} \Vdash \phi$, as $\mathcal{U} \Vdash \phi[\sigma]$ for all (equivalently, some) $\mathcal{U}$-assignments σ. What we have called the Satisfaction Principle in section 2 becomes manifest in the notions of satisfiability and validity: a formula ϕ is called satisfiable in $\mathcal{U}$ just in case there is a $\mathcal{U}$-assignment σ that is 1-1 on the variables occurring free in ϕ such that $\mathcal{U} \Vdash \phi[\sigma]$, and valid in $\mathcal{U}$ if satisfied by every variable assignment that is 1-1 on the free variables in ϕ.[47]

Next we define a function $\mathcal{F}^=$ that maps $\mathcal{L}_\mathcal{T}$-formulas ϕ into formulas $\mathcal{F}^=(\phi)$ of FOL$^=$. Again, the definition proceeds by recursion on ϕ.

(1) $\mathcal{F}^=(Rx_1 \dots x_n)$ is just $Rx_1 \dots x_n$.
(2) $\mathcal{F}^=(\mathrm{N}(\phi_0, \dots, \phi_n))$ is $\bigwedge_{i=0}^n \neg\mathcal{F}^=(\phi_i)$.
(3) $\mathcal{F}^=(\mathrm{N}(x\colon\phi))$ is $\forall x\,(\bigwedge_{i=1}^n x \neq y_i \supset \neg\mathcal{F}^=(\phi))$, where $y_1, \dots, y_n$ are precisely the variables occurring free in $\mathrm{N}(x\colon\phi)$.

That $\mathcal{F}^=$ adequately translates $\mathcal{T}$ into FOL$^=$ is the content of the following lemma (here $\models$ is the standard Tarskian satisfaction relation).

Lemma 3. For all models $\mathcal{U}$ and all $\mathcal{U}$-assignments σ:

(1) For all $\mathcal{L}_\mathcal{T}$-formulas ϕ, $\mathcal{U} \Vdash \phi[\sigma]$ if and only if $\mathcal{U} \models \mathcal{F}^=(\phi)[\sigma]$.
(2) For all $\mathcal{L}_\mathcal{T}$-sentences ϕ, $\mathcal{U} \Vdash \phi$ if and only if $\mathcal{U} \models \mathcal{F}^=(\phi)$.

The first part is easily proved by induction on ϕ, and the second part is a special case of the first.

It is also possible to translate FOL$^=$ into Tractarian logic. The following recursively defined translation $\mathcal{T}$ accomplishes this (we are assuming for simplicity's sake that FOL$^=$ is based on the primitives $\neg$, $\vee$, and $\forall$):

(1) $\mathcal{T}(x = x)$ is $\mathrm{N}(\mathrm{N}(Px, \mathrm{N}(Px)))$, where P is some fixed unary predicate symbol.

(2) $\mathcal{T}(x = y)$ is $\mathrm{N}(Sxy, \mathrm{N}(Sxy))$, provided x and y are typographically distinct variables. Here S is some fixed binary predicate symbol.

(3) $\mathcal{T}(Rx_1 \dots x_n)$ is just $Rx_1 \dots x_n$ for any non-equational atomic formula of FOL$^=$.

(4) $\mathcal{T}(\neg\phi)$ is $\mathrm{N}(\mathcal{T}(\phi))$.

(5) $\mathcal{T}(\phi \vee \psi)$ is $\mathrm{N}(\mathrm{N}(\mathcal{T}(\phi), \mathcal{T}(\psi)))$.

(6) $\mathcal{T}(\forall x\phi)$ is

$$\mathrm{N}\left(\mathrm{N}(\mathrm{N}(x\colon \mathrm{N}(\mathcal{T}(\phi)))), \mathrm{N}(\mathcal{T}(\phi_x[y_1])), \dots, \mathrm{N}(\mathcal{T}(\phi_x[y_n]))\right),$$

where $y_1, \dots, y_n$ are precisely the variables occurring free in $\forall x\phi$ (and care is taken that y_i does not become unintentionally bound in $\phi_x[y_i]$).

Since $x = y$ is translated as a contradiction, we cannot in general expect equivalence of ϕ and $\mathcal{T}(\phi)$ under *all* variable assignments. However, we have the following:

Lemma 4. For every structure $\mathcal{U}$, every FOL$^=$-formula ϕ and every $\mathcal{U}$-assignment σ that is 1-1 on the variables occurring free in ϕ, $\mathcal{U} \models \phi[\sigma]$ if and only if $\mathcal{U} \Vdash \mathcal{T}(\phi)[\sigma]$.

The proof uses mainly propositional logic and the substitution lemma of first-order logic.

By noting that every assignment is 1-1 on the empty set of variables, it follows immediately from lemma 4 that full translational equivalence holds for sentences of FOL$^=$:

Corollary. For all structures $\mathcal{U}$ and all FOL$^=$-sentences ϕ, $\mathcal{U} \models \phi$ just in case $\mathcal{U} \Vdash \mathcal{T}(\phi)$.

The discussion in §2.2 of the corresponding lemmas 1 and 2 applies here as well.

5. Domain Size and Tableau Calculi

We now develop proof procedures for the system of Tractarian logic described in §4. Admittedly, Wittgenstein is sometimes thought to have rejected the idea of a deductive system wholesale. This notion appears to be based on *Tractatus* 6.126-6.1271, where, however, he objects rather to the illusion, engendered by axiomatic systems, that there are some propositions "which are essentially primitive and others deduced from these" (6.127). Instead of axiomatic deduction, Wittgenstein advocates truth-tabular calculation (6.1203). Now tableau calculi are the natural generalization of truth-tables to predicate logic, so we take the proof procedures introduced below to be entirely Tractarian in spirit. With reference to complete tableau calculi such as ours, it is indeed true that "every tautology itself shows that it is a tautology" (6.127).

A tableau is essentially a systematic method of searching for structures that make a given formula true (corresponding to completed open branches in a tableau). Thus to prove a formula ϕ, one constructs a tableau for the negation of ϕ (that is, in our setting, $\mathrm{N}(\phi)$) and attempts to show that there is no way of making it true (that is, that every branch in the tableau closes). In the context of tableaux, there is a certain drawback to the syntactical set-up we have been using so far; more precisely, to using the same set of symbols for free and bound variables. Consider tableaux for $\mathrm{FOL}^=$: If a universal formula $\forall x\phi$ occurs on some open branch, one may append any instance of $\forall x\phi$ to the bottom of the branch. Now if the only singular terms are the variables $x, y, z, \ldots$, it can happen that one appends $\phi_x[y]$ where y becomes unintentionally bound in the new formula, e.g. if the universal formula is $\forall x(Px \wedge \exists yRxy)$. It is not much help in such a case to require renaming of bound variables, since one would have to retrace the branch upwards to the original universal formula and rename there, and possibly an unlimited number of times, too. For this reason, it is typographically convenient to distinguish between bound and free variables when dealing with tableaux. For the purposes of this section, we thus indicate free variables by $a, b, a_0, a_1, a_2, \ldots$ while continuing to use $x, y, x_0, x_1, x_2, \ldots$ for bound variables.[48]

As is well known, in the *Tractatus* Wittgenstein holds that the domain of objects is fixed, that the objects exist necessarily:

> Only if there are objects can there be a fixed form of the world. (2.026)
> The fixed, the existent and the object are one. (2.027)
> The object is the fixed, the existent; the configuration is the changing, the variable. (2.0271)

At the same time, he makes no commitment as to the size of this domain, except presumably that it must be non-empty (cf. 2.026). As long as it remains undetermined, however, whether the fixed domain is finite or infinite, it is not possible to provide a sound and complete proof procedure for Tractarian logic, because the logical status of sentences expressing "there are at least n objects" is then also undetermined. In the sequel, we outline tableau calculi for Tractarian logic under two alternative assumptions: that the domain is infinite (§5.2) and that it is of a fixed finite cardinality (§5.3). These would seem to be the two possibilities entertained by Wittgenstein. In §5.1, we also provide a calculus for the case where the domain of objects is allowed to vary arbitrarily (as has become standard in contemporary presentations of first-order logic).

The following rules are common to all our calculi.

Propositional N**-rule:**
If an open branch contains a formula of the form $\mathrm{N}(\phi_0, \ldots, \phi_n)$, then any $\mathrm{N}(\phi_i)$ may be appended to the bottom of the branch.

Propositional N-N**-rule:**
If an open branch contains a formula of the form $\mathrm{N}(\mathrm{N}(\phi_0, \ldots, \phi_n))$, branch out into $n + 1$ new branches by appending each of $\phi_0, \ldots, \phi_n$ to the bottom of the existing branch.

N-N**-Instantiation Rule:**
If an open branch B contains a formula of the form $\mathrm{N}(\mathrm{N}(x\!:\!\phi))$, and the free variables occurring on B but not in ϕ are precisely $a_1, \ldots, a_n$, then one may create $n+1$ new branches by appending to the bottom of B the formulas $\phi_x[a_1], \ldots, \phi_x[a_n], \phi_x[b]$, respectively, where b is a free variable not yet occurring on B.

Closure Rule:
Close any branch on which both ϕ and $\mathrm{N}(\phi)$ occur, where ϕ is any formula.

Of these rules, only the N-N-Instantiation Rule requires a soundness argument. Think of the formulas on a branch B as the truths we are committed to, and of the free variables occurring in formulas on B as representing the objects we have already admitted into our ontology. Now a commitment to $\mathrm{N}(\mathrm{N}(x\!:\!\phi))$ entails a commitment to $\phi_x[a]$ for some object a other than those mentioned in ϕ. This a could be one of the objects we have already admitted (this situation corresponds to the first n new branches generated by N-N-Instantiation), or else (i.e. if none of the objects we have already recognized is a witness) we must admit an additional object to serve as the existential witness (and the $(n + 1)$st new branch takes care of this possibility).

5.1. **Variable Domain Tractarian Logic.** We now describe a tableau calculus that generates just those formulas that are valid in every non-empty structure $\mathcal{U}$. While this is the contemporary notion of logical truth, it is not the Tractarian one, because, as we saw, Wittgenstein conceives of the domain of objects as being fixed once and for all. We nevertheless include this section in order to demonstrate that Tractarian logic can easily accommodate the present-day definition of logical validity.

In addition to the common rules enumerated above, variable domain Tractarian logic has the following

$\mathrm{N_V}$-Instantiation Rule:

If an open branch B contains a formula of the form $\mathrm{N}(x\!: \phi)$, then one may append the formula $\mathrm{N}(\phi_x[a])$ to the bottom of B, provided that the free variable a does not occur in ϕ and either (i) a occurs on B or (ii) no free variables whatsoever occur on B.[49]

In order to prove a formula ϕ, one begins a tableau with the formula $\mathrm{N}(\phi)$ and tries to close all the branches. If one succeeds, ϕ is a theorem of variable domain Tractarian logic. Soundness and completeness of this procedure follow easily from the corresponding results in (Wehmeier 2008).

Some informal remarks concerning soundness may nevertheless be in order here. If we are committed to the truth of $\mathrm{N}(x\!: \phi)$, we are *ipso facto* committed to $\mathrm{N}(\phi_x[a])$ for every object a in our ontology, unless that object is assigned to a free variable occurring in ϕ. This is the content of case (i) of $\mathrm{N_V}$-Instantiation. If we have not yet explicitly admitted any objects whatsoever, we simply observe that the domain is not allowed to be empty, so that one object a is always available for instantiation. This is case (ii).

5.2. **Infinite Domain Tractarian Logic.** Let U be an infinite set of objects. Our task is to devise a calculus for generating those formulas of $\mathcal{L}_\mathcal{T}$ that are valid in every structure $\mathcal{U}$ with domain U. By the Löwenheim-Skolem theorems, these are precisely those $\mathcal{L}_\mathcal{T}$-formulas valid in every infinite structure. This implies that the following straightforward modification of the tableau calculus for variable domain Tractarian logic does the job: To prove ϕ, begin the tableau with $\mathrm{N}(\phi)$ and apply the rules for variable domain Tractarian logic, plus after the nth rule application, for every $n \geq 2$, append a sentence expressing that there are at least n objects[50] to the end of every open branch. If every branch closes, ϕ has been proved. This procedure is sound and complete with respect to validity in all infinite structures.[51]

Our objective can be met in a more elegant and less roundabout way, however. Simply add to the common rules the following

N_{I}**-Instantiation Rule:**

If an open branch contains a formula of the form $\mathrm{N}(x\colon\phi)$, then one may append the formula $\mathrm{N}(\phi_x[a])$ to the bottom of the branch, provided that the free variable a does not occur in ϕ.

We again provide an informal argument for the soundness of this rule. A commitment to $\mathrm{N}(x\colon\phi)$ carries with it a commitment to $\mathrm{N}(\phi_x[a])$ for each object in our ontology except the finitely many objects $b_1, \ldots, b_n$ already referred to in $\mathrm{N}(x\colon\phi)$. Given that the domain is infinite, we know that objects not among $b_1, \ldots, b_n$ exist; hence we can do without the restrictions necessary in the case of variable domain logic.[52]

5.3. **Fixed Finite Domain Tractarian Logic.** Let U be a finite set with $n \geq 1$ members. A formula ϕ is valid over U if and only if it is valid in all models with domain U, if and only if it is valid in all models with a domain of cardinality n, if and only if it is a logical consequence in the standard sense of a first-order sentence expressing "there are exactly n objects". We can therefore obtain a calculus for Tractarian logic over a domain of size n by modifying the calculus for variable domain Tractarian logic as follows: To prove a formula ϕ, begin a variable domain logic tableau with the formula $\mathrm{N}(\phi, \mathrm{N}(\psi_n))$, where ψ_n is a sentence true in exactly the size-n domains.[53] If the tableau closes, $\mathrm{N}(\phi)$ is inconsistent with there being exactly n objects, and hence ϕ is valid in all models of size n.

We can, however, offer a slightly more elegant calculus. Add to the common rules the following two rules:

N_n**-Instantiation Rule:**

If an open branch contains a formula of the form $\mathrm{N}(x\colon\phi)$, then one may append the formula $\mathrm{N}(\phi_x[a])$ to the bottom of the branch, provided that the free variable a does not occur in ϕ and that the resulting branch does not contain more than n free variables.

n**-Closure Rule:**

Close every branch that contains more than n free variables.

Since the domain U only contains n objects, it is clear, by the Tractarian Satisfaction Principle, that a branch containing more than n free variables is unsatisfiable in U, whence the n-Closure Rule and the restriction on the N_n-Instantiation Rule. Note that no infinite branches are possible; hence the logic is decidable, as is its $\mathrm{FOL}^=$-counterpart.[54]

6. Conclusion

We have noted repeatedly that Wittgenstein's exposition of his logical proposals is neither as detailed nor as complete as one might wish. Some commentators have suggested that this fact could be indicative of a lack of interest in a formal logical system on Wittgenstein's part:

> Wittgenstein's remarks on notation are so vague and scattered that the idea that the development of a smooth-running formalized language was his aim requires, at the very least, some working out. (Floyd 2007, p. 195)

We have provided such a working out in this essay by giving "exact expression" (5.503) to precise rules that account for Wittgenstein's suggestions and examples. Indeed, we take the rigorous presentability of Tractarian Logic and its expressive equivalence with $\mathrm{FOL}^{=}$ to refute the contention that "Wittgenstein's remarks drastically limit the usual ways in which quantification theory is presented" (Floyd 2002, 326). This undermines claims to the effect that "a smooth-running system of pure logic holds no interest for Wittgenstein" and that "he has no interest in presenting a systematic way of deriving patterns of quantificational structure from other patterns of quantificational structure" (ibid.).

An obvious complaint against Wittgenstein's contention that "every proposition is the result of successive applications of the operator N'$(\bar{\xi})$ to the elementary propositions" (6.001) is that, in a fully explicit notation such as Geach's or the one employing the circumflex and the horizontal bar, it is revealed that Wittgenstein in effect uses two substantially different N-operators rather than just one: a quantificational, variable-binding operator $\mathrm{N}(x\colon \ldots)$ on the one hand, and a sentential operator $\mathrm{N}(\ldots, \ldots, \ldots)$ on the other, the latter being reducible to the binary Sheffer stroke. In the context of first-order logic, which is what we have focused on in this essay, such a view seems entirely reasonable.

First-order logic, however, is not in any obvious way a natural habitat for the N-operator; analyzing it in this environment may make appear as distinct what are really infelicitously carved-out slices of a unary concept. To use a geometrical analogy, the two branches of a hyperbola appear disconnected when considered as graphs in a plane. Nevertheless, they are in fact the result of intersecting *one* conical surface with a plane. Thus viewing the hyperbola in its natural, three-dimensional environment reveals a connectedness of its branches that was invisible in the plane. We suggest that the case of the N-operator may be similar.

In a system that includes the possibility of referring to propositional functions $f\hat{x}$ and to the classes $\overline{f\hat{x}}$ of all values of such functions, it is much less clear that the "quantificational" and the "sentential" uses of N really are categorially distinct. In either case, the operation of joint denial is applied to a class of propositions; it is just that, in the former case, N is applied to a class $\overline{f\hat{x}}$ specified with the help of a bound variable $\hat{x}$, and in the latter, without any variable-binding apparatus. It is therefore not the operation of joint denial itself that appears now as quantificational, now as sentential, but only the description of its arguments. Thus, in an appropriately rich logical system, it appears to be possible to reconcile these ostensibly distinct natures of N with Wittgenstein's insistence that every proposition can be represented as the result of iterated applications of the single operator N.

Notes

[1] *Tractatus* references are to (Wittgenstein 1922), i.e. the Ogden translation. We follow the usual practice of citing propositions by means of their assigned decimal numbers.

[2] Glock (1996, p. 222) argues that Wittgenstein's "account of the General Propositional Form"—the symbol introduced in proposition 6, encapsulating Wittgenstein's contention that all propositions can be represented by the successive application of the N-operator to elementary propositions—"is successful only if the number of elementary propositions is finite", and thus, presumably, only if the number of objects is finite. Anscombe (1959, pp. 136-137) comes to the same conclusion.

[3] A note on logical typography: When discussing formulas first introduced in a quotation, we usually employ the typographical conventions of the quotation's author. In our own exposition, we use standard contemporary notation. As a consequence, this paper features a variety of logical typographies. We believe that this solution is preferable to the mutilation of quotations by either tampering with their originators' conventions or overloading them with editorial annotations containing transcriptions.

[4] In Wittgenstein's examples, a and b may equivalently be construed as names, in which case the Satisfaction Principle amounts to the requirement that no two names corefer.

[5] It makes no difference to either of the strongly exclusive interpretations whether we include the second conjunct within the scope of the existential quantifier: Either strongly exclusive reading assigns the same truth conditions to the formula $\exists x Px \wedge Pa$ as it does to $\exists x(Px \wedge Pa)$. The weakly exclusive interpretation, by contrast, treats the two formulas differently. On the former, it agrees with the narrow, on the latter with the broad strongly exclusive reading.

[6] This formula says that if anything other than a is F, then a is F, and that there is at most one object distinct from a that is F. Such is the case if and only if either nothing is F or a and at most one other object are F.

[7] The remaining examples in 5.532 and 5.5321 are consistent with each of the three readings; in particular, they cannot decide between the broad and the narrow strongly exclusive interpretations, because they do not contain any free variables.

[8]Mnemonics: Translation functions of the $\mathcal{R}$ type translate into standard ("Russellian") notation, the source language being given by sub- and superscripts: "w" for weakly exclusive, "s" for strongly exclusive, "b" for broad, "n" for narrow. $\mathcal{W}$ translates from standard into weakly exclusive, $\mathcal{S}_b$ into broad, and $\mathcal{S}_n$ into narrow strongly exclusive notation.

[9]See §2.3 for further discussion.

[10]$\phi_x[y]$ is the result of replacing all free occurrences of x in ϕ by y. We are assuming that no substituted y_i becomes unintentionally bound in $\phi_x[y_i]$; if necessary, bound variables must be renamed prior to substitution.

[11]We're assuming that x is not itself in V; otherwise consider a suitable variant $\exists z \phi_x[z]$ instead of $\exists x \phi$, and similarly for the universal quantifier.

[12]Here, too, we assume that x is not among $y_1, \ldots, y_n$; otherwise we first rename the bound variable x.

[13]It follows from our translation procedures that consecutive like quantifiers commute under both strongly exclusive exclusive readings, just as they do in FOL$^=$. Under the weakly exclusive readings, $\exists x \exists y \phi$ is equivalent to $\exists y \exists x \phi$ if either x and y both occur free, or both fail to occur free, in ϕ; otherwise the equivalence can fail, as witnessed by $\exists x \exists y Py$ (where no variable occurs free in either quantifier's scope) and $\exists y \exists x Py$ (where y occurs free in the scope of $\exists x$). Call a quantifier occurrence vacuous if its bound variable does not occur in its scope. In FOL$^=$, vacuous quantifier occurrences can be deleted without changing the truth conditions of a formula. This is not always the case in any of the three exclusive interpretations: These all require at least two objects in order to make $\exists y \exists x Fy$ true, but allow for $\exists y Fy$ to be true in a one-element domain.

[14]Cf. §4 for an alternative approach using an independent semantics for a language employing the weakly exclusive interpretation and the N-operator.

[15]In contemporary presentations of predicate logic, this result is often referred to as the eliminability of constant symbols. See (Boolos, Burgess, and Jeffrey 2007, pp. 255-257) for details.

[16]The lemma amounts to the claim that ϕ, $\mathcal{R}_w(\mathcal{W}(\phi))$, $\mathcal{R}_s^b(\mathcal{S}_b(\phi))$, and $\mathcal{R}_s^n(\mathcal{S}_n(\phi))$ are all equivalent in FOL$^=$. This is so even though these formulas are not in general syntactically identical.

[17]For proofs of lemmas 1 and 2, the reader is referred to (Wehmeier 2004) or (Wehmeier 2008) for the weakly exclusive interpretation, and to (Wehmeier 2009) for the broad strongly exclusive interpretation; the case of the narrow strongly exclusive interpretation follows by simple modifications to the broad strongly exclusive case.

[18]This is proved in (Wehmeier 2008), where it is also shown that supplementing the weakly exclusive idiom with a Fregean co-denotation relation $\equiv$ restores full expressive equivalence, even if only names (but no bound variables) are permitted to flank the triple bar.

[19]Fogelin (1987, p. 84), McGray (2006, p. 159), and Frascolla (2007, pp. 144-145) also conclude that the formulas mentioned in 5.534 have no counterparts in Wittgenstein's alternative notation.

[20]Compare Frege's (1969, pp. 141-142) use of *Scheingedanke* in the 1897 fragment "Logik", by which he does not mean something that fails to be a *Gedanke*, but rather a *Gedanke* that is defective in a particular way (to wit, in that it lacks a truth value).

[21]Could one block this move by insisting that $b = a$ is not a proposition in the sense intended at 4.465? That seems implausible, for if $b = a$ fails to be a proposition, $fb \,.\, b = a$ and therefore $(\exists x).fx \,.\, x = a$ should also not be propositions—but Wittgenstein explicitly states that the latter says the same as fa at 5.441.

[22]Ramsey's descriptions of these sessions appear in (McGuinness 2008, pp. 139-140).

[23]The translation provided by Wittgenstein is also incompatible with the broad strongly exclusive reading. Cf. (Wehmeier 2009) for a discussion of Ramsey's initial misunderstanding of Wittgenstein's convention.

[24]The same restrictions apply, of course, to universally quantified formulas.

[25]This is equivalent to the characterization of the weakly exclusive reading provided on p. 325 of (Floyd 2002): "[. . .] the 'weakly exclusive' reading would apply only to variables within the scope of a (sequence of) quantifier(s)", in such a way that "bound variables occurring within sequences of more than one quantifier (whether in a subformula or in a sentence) are interpreted as having ranges of significance that are restricted as we move from left to right through the sentence."

[26]Cf. (Floyd 2002, p. 325): "[. . .] the 'strongly exclusive' reading would demand that every distinct bound variable throughout the whole sentence have a range restricted by all the previously occurring quantifiers."

[27]Independently of *Tractatus* exegesis, the roughshod interpretation is questionable because it constitutes a radical break with the standard conception of operator scope. Consider the formula $\forall xFx \wedge \exists yGy$. How should we read it, according to the roughshod interpretation? Presumably the range of the bound variable y must not contain any value we assign to the bound variable x. Thus the formula should translate into standard notation as something like $\forall xFx \wedge \exists y(\neg x = y \wedge Gy)$. But now x occurs free in the translation, which cannot be right, because the original formula was a sentence and thus not dependent on any choice of value for particular variables. The only way to avoid this problem is to tacitly extend the scope of the universal quantifier $\forall x$ over the entire formula, that is, to translate $\forall xFx \wedge \exists yGy$ as $\forall x(Fx \wedge \exists y(\neg x = y \wedge Gy))$. This means, in effect, to abandon the quantifier-variable mechanism for formulas written in roughshod notation, for the scope of x in $\forall xFx \wedge \exists yGy$ is now the entire formula, and not just its first conjunct.

[28]The criticism of Wittgenstein's convention in (Carnap 1937, p. 50) appears to be based on the same assumption.

[29]Bogen makes a similar criticism in correspondence with Fogelin (reported in Fogelin 1983, p. 143), arguing that $(x)(Fx) \supset (\exists y)(Fy)$ "ought to count as a logical truth" but does not in Wittgenstein's notation. Fogelin agrees with Bogen's assessment, claiming that the above sentence "does not formulate a logical truth in Wittgenstein's system" (ibid.). Of course it cannot be reasonably expected that every valid formula in the Russellian notation will also be valid in the Wittgensteinian notation, but under the weakly exclusive and both strongly exclusive interpretations, $(x)(Fx) \supset (\exists y)(Fy)$ in fact *is* a logical truth, because the scopes of x and y do not overlap. Bogen and Fogelin thus show that they attribute the roughshod reading to Wittgenstein, which we have argued cannot have been what Wittgenstein intended.

[30]Landini only considers the second instance listed above. He reads 5.532 as suggesting that Wittgenstein "takes a quantifier to exclude another, when and only when it is in the scope of the other" (2007, p. 254). This is the strongly exclusive

interpretation, of course, which we have seen to be compatible with the text only in its narrow version. As we have argued, it is more likely that Wittgenstein actually intended the weakly exclusive interpretation.

[31]The superscripted quantifiers of Landini's calculus obviously represent a drastic departure from Wittgenstein's proposal, and in any case appear to be but a thinly disguised notational variant of certain $\mathrm{FOL}^{=}$-expressions: In essence, Landini's $(\forall x^{y_1,\ldots,y_n})\phi$ is just another way of writing $(\forall x)(\bigwedge_{i=1}^{n} x \neq y_i \supset \phi)$, and likewise for his $(\exists x^{y_1,\ldots,y_n})\phi$ and the $\mathrm{FOL}^{=}$-formula $(\exists x)(\bigwedge_{i=1}^{n} x \neq y_i \wedge \phi)$. As it stands, the calculus is unsound, due to axiom schema 7 (Landini 2007, p. 259): $(\forall x^{y,z_1,\ldots,z_n})A \supset (\forall x^{z_1,\ldots,z_n})A$, where y doesn't occur free in A (in personal communication, Landini has proposed $(\forall x^{z_1,\ldots,z_n})A \supset (\forall x^{y,z_1,\ldots,z_n})A$ as a replacement). Apart from these issues, assessment of Landini's calculus is made difficult by the absence of a semantics for his language, of soundness and completeness proofs, and of translation procedures between his notation and $\mathrm{FOL}^{=}$ (Landini only provides some examples for translating from $\mathrm{FOL}^{=}$ into his notation).

[32]Note that no apostrophe occurs in this remark, as opposed to 6.001, quoted in the introduction. In the Ogden translation (Wittgenstein 1922), 6.001 is the only place where an apostrophe occurs after "N". This occurrence is mysteriously missing from both the German and English texts of the Pears-McGuinness edition (Wittgenstein 1961b). We take the apostrophe to have the same meaning as in *Principia*, where it is explained as indicating function application (p. 245). See (Sundholm 1992, p. 61) for a comparison of the use of this device in the *Tractatus* and *Principia*. The parentheses following N already indicate application of a function to an argument, so that the apostrophe is systematically expendable. We therefore do not use it in our own exposition.

[33] Cf. p. 15 of *Principia*: "Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all."

[34]This claim is encapsulated in the symbol $[\overline{p}, \overline{\xi}, \mathrm{N}(\overline{\xi})]$, which Wittgenstein calls "the general form of proposition" (6). One of Wittgenstein's motivations for using a notation with a single logical operator appears to be that formal inferences can thereby be displayed in a more perspicuous and intuitive manner (cf. 5.1311). This is born out in section 5, where we provide tableau calculi for Tractarian logic that employ a remarkably small number of simple inference rules.

[35]The full Fogelin-Geach exchange takes place in (Fogelin 1976), (Geach 1981), (Fogelin 1982), (Geach 1982), and (Fogelin 1987). A significant literature has emerged in response to this dispute, nearly uniformly in opposition to Fogelin's position. See (Soames 1983), (Sundholm 1992), (Varga von Kibéd 1993), (Miller 1995), (Cheung 2000), (Jacquette 2001), (Floyd 2002), (McGray 2006), and (Landini 2007).

[36]Cheung (2000, p. 248) and Morris (2008, p. 377 n. 8) also make these points about Fogelin's notation.

[37]Geach includes an umlaut above the first x. Since this symbol is superfluous we do not use it.

[38]Fogelin's reaction confirms that he does not believe that there can be complex propositional functions in N-notation. He contrasts Geach's $\mathrm{N}(x\colon \mathrm{N}(fx))$ with the application of N to the Russellian propositional function $\sim fx$:

> [...] the expression $\mathrm{N}({\sim}fx)$ will generate propositions from a set of (possibly) infinitely many propositions through a single application of the operation N. Here the symbol for negation is treated as a constituent of the propositional function used to generate the set of propositions. By way of contrast, the inner-most "*N*" in Geach's $\mathrm{N}(x\colon \mathrm{N}(fx))$ is not a constituent of a propositional function at all and to think otherwise is to misunderstand its role entirely. (1987, p. 80)

On the contrary, as we have argued, the role of Geach's notation is *precisely* to allow us to convert a proposition that is the result of applying the N-operator into a propositional function. Fogelin's refusal to allow N to be a constituent of a propositional function shows that he does not admit the possibility of complex propositional functions, a position which is refuted by 3.315. Like us, Miller (1995) argues that Geach's notation serves the purpose of describing propositional variables with complex propositions as values, but Cheung (2000, pp. 250-251) claims that this reading of Geach's notation would make the construction of propositions such as $\mathrm{N}(x\colon \mathrm{N}(fx))$ involve two operations: the N-operator plus a second operation that converts propositions to propositional variables. Cheung's criticism fails to follow Wittgenstein's use of the term "operation" in the *Tractatus*: "A function cannot be its own argument, but the result of an operation can be its own basis" (5.251). For Wittgenstein, an operation can always be iterated; using current terminology, we would say that an operation is a function whose range is a subset of its domain. The transition from a proposition to a propositional variable therefore does not qualify as an operation for Wittgenstein, because the resulting propositional variable cannot subsequently serve as a basis for this function. Regardless, such a transition is clearly sanctioned by 3.315. See §6 for further discussion.

[39]As the purpose of Geach's notation is to express propositional functions, the expression $\mathrm{N}(x\colon \mathrm{N}(fx))$ cannot just be shorthand for $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb), \mathrm{N}(fc), \ldots)$, because that would mean that propositional variables obtained by the second method of 5.501 are merely abbreviations for enumerated lists. If this were the case, then the first and second methods of description of 5.501 would actually be identical. According to Wittgenstein, however, they are quite different, for 5.51 shows us that applying N to an enumerated list is equivalent to the joint negation of all the listed propositions, while 5.52 shows that the result of applying N to a propositional variable derived from a propositional function corresponds to a *quantified* proposition. As we argue above, this is because propositional functions exhibit features of the "symbolism of generality" described at 5.522. Note that the fact that a propositional variable can have an infinite number of propositions as values while an enumerated list must be finite is not the reason why the result at 5.52 is equivalent to a quantified propositions while those of 5.51 are not. In Tractarian logic, generality results from applying N to a class of values of a propositional function; even if the domain is finite, $\mathrm{N}(\bar{\xi})$ is equivalent to ${\sim}(\exists x)fx$ if $\bar{\xi}$ is the class of all values of the propositional function fx. Although he argues against Fogelin's charge of expressive incompleteness, Varga von Kibéd (1993, p. 80-81) treats every propositional variable as an abbreviation for an enumerated list of propositions, and thus follows Fogelin's conflation of the first and second methods of 5.501. Jacquette (2001, p. 197) similarly argues that general propositions can result from applying N to enumerated lists.

[40]Since this dispute hinges on interpreting a translated passage in subtly different ways, Wittgenstein's original formulation should be consulted. In the German text he speaks of "Resultate der successiven Anwendung einer endlichen Anzahl von Wahrheitsoperationen" [results of the successive application of a finite number of truth-operations], and not, as Fogelin's reading would require, of "Resultate einer endlichen Anzahl successiver Anwendungen von Wahrheitsoperationen" [results of a finite number of successive applications of truth-operations]. Schroeder (2006, p. 81 fn. 30) reads 5.32 as we do, and gives 5.2521 as evidence that Wittgenstein carefully distinguished between the number of truth-operations and the number of applications of an operation:

> The repeated application of an operation to its own result I call its successive application ("O' O' O' a" is the result of the threefold successive application of "O' ξ" to "a").

This passage also shows that Fogelin's concern about *successive* applications of N is unfaithful to Wittgenstein's use of this term in the *Tractatus*, as Fogelin argues that Geach's notation

> [...] also violates the demand for successiveness. If the set of base propositions is infinite, then nothing will count as the immediate predecessor of the final application of the operation N in the construction of a universally quantified proposition. (1987, p. 81)

Fogelin construes $\mathrm{N}(x\colon \mathrm{N}(fx))$, Geach's suggested symbolization of $(\forall x)fx$, as merely shorthand for $\mathrm{N}(\mathrm{N}(fa), \mathrm{N}(fb), \mathrm{N}(fc), \ldots))$, and then points out that in the latter expression, an infinite number of applications of N must be completed before the outermost N-operator can be applied. Since this final N-operation has no immediate predecessor, Fogelin argues that the N-operations in this expression cannot be *successively* applied. He therefore takes Wittgenstein's use of "successive" in 5.32 to indicate a requirement that the applications of N in any sentence of Tractarian logic must be to a well-ordered series of propositions with a last element. But 5.2521 shows that Wittgenstein associates successiveness with the iterated application of an operation (i.e. the application of the operation to its own results), and not with such a well-ordered series of individual applications of an operation to a list of propositions. See Cheung (2000, pp. 251-254) for a convincing construal of the general form of proposition as a partially ordered series of propositions resulting from successive applications of the N-operator.

[41]Fogelin admits this as an unlikely "alternative reading" (p. 73) in (1976), but drops any such admission when revisiting the topic in (1987).

[42]The *Principia* passage just quoted illustrates that Russell himself was somewhat casual when it came to distinguishing between an "ambiguous value" of a propositional function and the propositional function itself. A few pages later, the authors of *Principia* downplay the significance of the circumflex notation when they explain that they "have found it convenient and possible—except in the explanatory portions—to keep the explicit use of symbols of the type "$\phi\hat{x}$," either as constants [e.g. $\hat{x} = a$] or as real variables, almost entirely out of this work" (*Principia*, p. 20; insertion in square brackets in the original). Thus Wittgenstein was no more cavalier with respect to the circumflex notation than his teacher.

[43]For the sake of readability, we do not enclose overlined expressions in parentheses.

[44]That Wittgenstein was familiar and comfortable with representing propositional functions by means of the circumflex notation is evidenced by *Prototractatus* 5.3321: "And the proposition '*Only* one x satisfies $F(\hat{x})$', will read '$(\exists x).Fx : \sim(\exists x, y).Fx.Fy$'" (Wittgenstein 1971, p. 183). Compare *Tractatus* 5.5321, where $F(\hat{x})$ is replaced by $f(\,)$.

[45]τ is an x-variant of σ if τ and σ agree on what they assign to all variables except possibly x.

[46]At 5.501, Wittgenstein stipulates that "the line over the variable indicates that it stands for *all* its values" (emphasis added). When the constraints of the weakly exclusive interpretation are added to N-logic, this stipulation must be amended. If ξ is specified by means of the function $\varphi(\hat{x})$, the expression $\overline{\varphi(\hat{x})}$ stands for the class of all values of $\varphi(\hat{x})$ except for $\varphi(b)$, where b is the value of any variable occurring free in $\varphi(\hat{x})$. Thus in Geach's notation, $(x\!:\!\varphi(x))$ excludes all such values $\varphi(b)$.

[47]Note that validity does not in general imply satisfiability: The formula $\mathrm{N}(\mathrm{N}(Rxy, \mathrm{N}(Rxy)))$, which corresponds to $Rxy \vee \neg Rxy$, is not satisfiable in a one-element domain, because there are no variable assignments in such a domain that are 1-1 on $\{x, y\}$. For this very reason, the formula is valid in one-element domains, as it is indeed in all domains. For $\mathcal{L}_{\mathcal{T}}$-sentences, however, the implication from validity to satisfiability continues to hold.

[48]Cf. (Smullyan 1968) for a similar syntactical set-up.

[49]Note that if there are free variables occurring on B, but they all also occur in $\mathrm{N}(x\!:\!\phi)$, the $\mathrm{N_V}$-Instantiation Rule cannot be applied.

[50]For instance, the following sentence χ_n will do:
$\mathrm{N}(\mathrm{N}(x_1 : \ldots \mathrm{N}(\mathrm{N}(x_n : \mathrm{N}(\mathrm{N}(\top(x_1)), \ldots, \mathrm{N}(\top(x_n))))) \ldots)).$

[51]This can be seen by simple modifications of the argument given by Smullyan (1968, 63-65).

[52]Straightforward modifications of the completeness proof for variable domain logic in (Wehmeier 2008) can be used to establish completeness for infinite domain Tractarian logic.

[53]We may take $\mathrm{N}(\mathrm{N}(\chi_n), \chi_{n+1})$ as ψ_n, where χ_n is as defined in note 50.

[54]Again, completeness follows by a simple modification of the proof in (Wehmeier 2008).

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