ACTUALITY IN PROPOSITIONAL MODAL LOGIC

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Abstract. We show that the actuality operator A is redundant in any propositional modal logic characterized by a class of Kripke models (respectively, neighborhood models). Specifically, we prove that for every formula ϕ in the propositional modal language with A, there is a formula ψ not containing A such that ϕ and ψ are materially equivalent at the actual world in every Kripke model (respectively, neighborhood model). Inspection of the proofs leads to corresponding proof-theoretic results concerning the eliminability of the actuality operator in the actuality extension of any normal propositional modal logic and of any "classical" modal logic. As an application, we provide an alternative proof of a result of Williamson's to the effect that the compound operator $\mathsf{A}\Box$ behaves, in any normal logic between T and $\mathsf{S5}$, like the simple necessity operator $\Box$ in $\mathsf{S5}$.

1. Introduction

We prove by semantic means that the actuality operator is eliminable in any propositional modal logic characterized by a class of Kripke models. It follows that adding this operator to any such logic does not increase the logic's expressive power (with respect to real-world truth). This situation contrasts with the non-eliminability of the actuality operator in quantified modal logics: see e.g. [5] and [11].

On the proof-theoretic side, we distill from our semantic argument a simple way of axiomatizing the actuality extension S_A of any normal propositional modal logic S. In any such logic S_A, the actuality operator is provably eliminable. Our axiomatizations might be profitably compared to the systems discussed in [4] (see also [1]).

Analogous model-theoretic and proof-theoretic results for logics characterized by classes of neighborhood models are also provided.

Existing proof-theoretic results by Crossley and Humberstone [3] and Hazen [6] imply that the addition of the actuality operator to the propositional modal logic $\mathsf{S5}$ engenders no increase in expressive power. More precisely, their results show that for every formula ϕ in the propositional modal language with the actuality operator, there is a formula ψ in the language without actuality such that ϕ and ψ are materially equivalent at the actual world of every $\mathsf{S5}$ Kripke model. Stephanou [10] establishes the eliminability of actuality operators in systems weaker than $\mathsf{S5}$ by proof-theoretic means. He considers a language in which all modal and actuality operators are indexed with natural numbers, however, so the linguistic setting is

Author names are listed alphabetically. Sections 2 and 3 contain joint work by KW and BR, with a crucial suggestion from AH. The remainder of the paper consists of joint work by AH and KW.

somewhat different from ours, and he does not consider logics with neighborhood semantics. Nevertheless, Stephanou's syntactic results MT5 and MT6 are similar to our model-theoretic Lemma 3.5 and Theorem 3.6.

The appendix contains an alternative proof, based on our A-elimination algorithm, of a result first obtained by Williamson [12]: the compound operator $\mathsf{A}\Box$ behaves, in any normal logic between T and $\mathsf{S5}$, like the simple necessity operator $\Box$ in $\mathsf{S5}$.

2. Definitions and Terminology

The primitive symbols of the language $\mathcal{L}_\mathsf{A}$ are the propositional variables, written $p, q, r, \ldots$; the propositional constants $\bot$ and $\top$; the truth-functional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$; the box $\Box$; and the actuality operator A. The $\mathcal{L}_\mathsf{A}$-formulas are generated inductively from the propositional variables, as well as $\bot$ and $\top$, as usual; in particular, whenever ϕ is an $\mathcal{L}_\mathsf{A}$-formula, so are $\mathsf{A}\phi$ (the *actualization* of ϕ) and $\Box\phi$ (the *necessitation* of ϕ). We define empty disjunctions (i.e. disjunctions $\bigvee_{i=k}^{n} \phi_i$ with $k > n$) to be $\bot$, and empty conjunctions (i.e. conjunctions $\bigwedge_{i=k}^{n} \phi_i$ with $k > n$) to be $\top$.

The language $\mathcal{L}$ is $\mathcal{L}_\mathsf{A}$ minus the actuality operator (i.e. the usual language of propositional modal logic); hence every $\mathcal{L}$-formula is an $\mathcal{L}_\mathsf{A}$-formula. An $\mathcal{L}_\mathsf{A}$-formula is an A-*free formula* if it contains no occurrences of A, i.e. if it is an $\mathcal{L}$-formula.

Any propositional variable p, any actualization $\mathsf{A}p$ of a propositional variable p, any necessitation $\Box\psi$ of an A-free formula ψ, and any actualization $\mathsf{A}\Box\theta$ of a necessitation of an A-free formula θ is called an A-*atom*. An A-*literal* is an A-atom or the negation of an A-atom.

A *Kripke model* $\mathbf{K}$ is a quadruple $(W, @, R, V)$ consisting of a set W of worlds, a distinguished element $@$ of W called the *actual world* of $\mathbf{K}$, a relation $R \subseteq W \times W$ called the *accessibility relation*, and a *verification relation* V between worlds and propositional variables.

Truth of an $\mathcal{L}_\mathsf{A}$-formula ϕ *at a world* $w \in W$ in such a Kripke model $\mathbf{K}$, $\mathbf{K} \models_w \phi$, is defined inductively as usual. In particular, $\mathbf{K} \models_w p$ if and only if wVp for propositional variables p; $\mathbf{K} \not\models_w \bot$; $\mathbf{K} \models_w \top$; $\mathbf{K} \models_w \Box\phi$ if and only if $\mathbf{K} \models_v \phi$ for all $v \in W$ such that wRv; and $\mathbf{K} \models_w \mathsf{A}\phi$ if and only if $\mathbf{K} \models_@ \phi$. We say that ϕ is *true in* $\mathbf{K}$, $\mathbf{K} \models \phi$, if $\mathbf{K} \models_@ \phi$.

Let $\mathcal{C}$ be a class of Kripke models. The $\mathcal{L}_\mathsf{A}$-formulas ϕ and ψ are *real-world equivalent over* $\mathcal{C}$ if for every Kripke model $\mathbf{K} \in \mathcal{C}$, $\mathbf{K} \models \phi \leftrightarrow \psi$ (that is, $\mathbf{K} \models_@ \phi \leftrightarrow \psi$). The $\mathcal{L}_\mathsf{A}$-formulas ϕ and ψ are *strictly equivalent over* $\mathcal{C}$ if for every Kripke model $\mathbf{K} \in \mathcal{C}$ and every world w of $\mathbf{K}$, $\mathbf{K} \models_w \phi \leftrightarrow \psi$. We say that ϕ and ψ are real-world (respectively, strictly) equivalent *tout court* if they are real-world (respectively, strictly) equivalent over the class of all Kripke models. Note that the formulas $\mathsf{A}p$ and p, for instance, are real-world equivalent but not strictly equivalent.

3. Actuality in Kripke Semantics

Lemma 3.1. *For any $\mathcal{L}_\mathsf{A}$-formulas ϕ and ψ:*

(1) $\mathsf{A}\neg\phi$ *is strictly equivalent to* $\neg\mathsf{A}\phi$.
(2) $\mathsf{A}\mathsf{A}\phi$ *is strictly equivalent to* $\mathsf{A}\phi$.
(3) $\mathsf{A}\neg\mathsf{A}\phi$ *is strictly equivalent to* $\neg\mathsf{A}\phi$.
(4) $\mathsf{A}(\phi \vee \psi)$ *is strictly equivalent to* $\mathsf{A}\phi \vee \mathsf{A}\psi$.

Proof. Easy exercise. $\square$

Lemma 3.2. *For any $\mathcal{L}_{\mathsf{A}}$-formulas ϕ and ψ, $\Box(\mathsf{A}\phi \vee \psi)$ is strictly equivalent to $\mathsf{A}\phi \vee \Box\psi$.*

Proof. Let $\mathbf{K} = (W, @, R, V)$ and $v \in W$. The following are equivalent:

- $\mathbf{K} \models_v \Box(\mathsf{A}\phi \vee \psi)$
- for all w with vRw, $\mathbf{K} \models_w \mathsf{A}\phi \vee \psi$
- for all w with vRw, $\mathbf{K} \models_@ \phi$ or $\mathbf{K} \models_w \psi$
- $\mathbf{K} \models_@ \phi$ or for all w with vRw, $\mathbf{K} \models_w \psi$
- $\mathbf{K} \models_v \mathsf{A}\phi$ or $\mathbf{K} \models_v \Box\psi$
- $\mathbf{K} \models_v \mathsf{A}\phi \vee \Box\psi$

$\square$

Corollary 3.3. *For any $\mathcal{L}_{\mathsf{A}}$-formula ϕ, the formula $\Box\mathsf{A}\phi$ is strictly equivalent to $\mathsf{A}\phi \vee \Box\bot$.*

Proof. The formula $\mathsf{A}\phi$ is truth-functionally, and thus strictly, equivalent to the disjunction $\mathsf{A}\phi \vee \bot$. $\square$

Lemma 3.2 has a simpler proof in the case of a logic characterized by a class of Kripke models over serial frames (i.e. frames with the property that for every world v there is a world w such that vRw), since every world in every such Kripke model verifies $\neg\Box\bot$. For such logics the following result, which constitutes a strengthening of Corollary 3.3, is obvious.

Lemma 3.4. *Let ϕ be an $\mathcal{L}_{\mathsf{A}}$-formula. Then $\Box\mathsf{A}\phi$, $\mathsf{A}\phi$, and $\neg\Box\neg\mathsf{A}\phi$ are all strictly equivalent over any class $\mathcal{C}$ of serial Kripke models.*

This, together with the implication (which holds at all worlds of all models) from $\Box(\theta \vee \chi)$ to $\neg\Box\neg\theta \vee \Box\chi$, yields an alternative proof of Lemma 3.2 when restricted to any class of serial Kripke models: At every world of a serial Kripke model, $\Box(\mathsf{A}\phi \vee \psi)$ implies $\neg\Box\neg\mathsf{A}\phi \vee \Box\psi$, which is strictly equivalent to $\mathsf{A}\phi \vee \Box\psi$ by Lemma 3.4, which, again by Lemma 3.4, implies $\Box(\mathsf{A}\phi \vee \psi)$, so that the strict equivalence of $\Box(\mathsf{A}\phi \vee \psi)$ with $\mathsf{A}\phi \vee \Box\psi$ follows.

Lemma 3.5. *Every $\mathcal{L}_{\mathsf{A}}$-formula ϕ is strictly equivalent to a truth-functional combination of A-atoms.*

Proof. We proceed by induction on ϕ. The non-trivial cases are those where ϕ is of the form $\mathsf{A}\psi$ or of the form $\Box\psi$.

Consider first the case where ϕ is $\mathsf{A}\psi$. By induction hypothesis, ψ is strictly equivalent to a truth-functional combination of A-atoms. By the conjunctive normal form theorem of propositional logic, ψ is then strictly equivalent to an $\mathcal{L}_{\mathsf{A}}$-formula of the form $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} L_{ij}$, where each L_{ij} is an A-literal.

Now let $\mathbf{K} = (W, @, R, V)$ be any Kripke model, and let $w \in W$. The following are equivalent:

- $\mathbf{K} \models_w \mathsf{A}\psi$
- $\mathbf{K} \models_@ \psi$
- $\mathbf{K} \models_@ \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} L_{ij}$
- for every $i \in \{1, \ldots, n\}$ there is a $j \in \{1, \ldots, m_i\}$ such that $\mathbf{K} \models_@ L_{ij}$
- for every $i \in \{1, \ldots, n\}$ there is a $j \in \{1, \ldots, m_i\}$ such that $\mathbf{K} \models_w \mathsf{A}L_{ij}$

- $\mathbf{K} \models_w \bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \mathsf{A}L_{ij}$

Thus $\mathsf{A}\psi$ is strictly equivalent to $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \mathsf{A}L_{ij}$. We now observe that if L_{ij} is a propositional variable or a necessitation of an A-free formula, $\mathsf{A}L_{ij}$ is itself an A-atom. If L_{ij} is of the form $\neg p$ or $\neg\Box\theta$ with A-free θ, then $\mathsf{A}L_{ij}$ is strictly equivalent, by Lemma 3.1(1), to $\neg\mathsf{A}p$ or $\neg\mathsf{A}\Box\theta$, both of which are negations of A-atoms. If L_{ij} is of the form $\mathsf{A}p$ or $\mathsf{A}\Box\theta$ with A-free θ, then $\mathsf{A}L_{ij}$ is strictly equivalent, by Lemma 3.1(2), to $\mathsf{A}p$ or $\mathsf{A}\Box\theta$, which are A-atoms. If, finally, L_{ij} is of the form $\neg\mathsf{A}p$ or $\neg\mathsf{A}\Box\theta$ with A-free θ, then $\mathsf{A}L_{ij}$ is strictly equivalent, by Lemma 3.1(3), to $\neg\mathsf{A}p$ or $\neg\mathsf{A}\Box\theta$, both of which are negations of A-atoms.

Now replace every $\mathsf{A}L_{ij}$ in $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} \mathsf{A}L_{ij}$ that is not already an A-literal by the strictly equivalent A-literal as just discussed. This clearly preserves strict equivalence. The result is a truth-functional combination of A-atoms.

We now turn to the case where ϕ is of the form $\Box\psi$. As before, ψ is strictly equivalent to a formula $\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} L_{ij}$, where each L_{ij} is an A-literal. Then $\Box\psi$ is strictly equivalent to $\Box\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} L_{ij}$, and since the box distributes over conjunctions, $\Box\psi$ is strictly equivalent to $\bigwedge_{i=1}^n \Box\bigvee_{j=1}^{m_i} L_{ij}$. It thus suffices to show that any formula of the form $\Box\bigvee_{j=1}^{m} L_j$, with each L_j an A-literal, is strictly equivalent to a truth-functional combination of A-atoms. By relabeling if necessary we may assume that $L_1, \ldots, L_k$ contain A, while $L_{k+1}, \ldots, L_m$ are A-free. For $j \in \{1, \ldots, k\}$, let L'_j be $\mathsf{A}\neg p$, if L_j is $\neg\mathsf{A}p$, let it be $\mathsf{A}\neg\Box\theta$, if L_j is $\neg\mathsf{A}\Box\theta$, and let it be just L_j otherwise. By Lemma 3.1(1) $\Box\bigvee_{j=1}^{m} L_j$ is strictly equivalent to $\Box(\bigvee_{j=1}^{k} L'_j \vee \bigvee_{j=k+1}^{m} L_j)$. For any $j \in \{1, \ldots, k\}$, we can write L'_j as $\mathsf{A}L''_j$ with A-free L''_j because each such L'_j begins with A. By Lemma 3.1(4), $\Box(\bigvee_{j=1}^{k} L'_j \vee \bigvee_{j=k+1}^{m} L_j)$ is strictly equivalent to $\Box(\mathsf{A}\bigvee_{j=1}^{k} L''_j \vee \bigvee_{j=k+1}^{m} L_j)$. By Lemma 3.2, this is strictly equivalent to $\mathsf{A}\bigvee_{j=1}^{k} L''_j \vee \Box\bigvee_{j=k+1}^{m} L_j$. This is strictly equivalent to $\bigvee_{j=1}^{k} L_j \vee \Box\bigvee_{j=k+1}^{m} L_j$ by Lemma 3.1(4). As desired, this last formula is a truth-functional combination of A-atoms. $\Box$

Theorem 3.6. *Every $\mathcal{L}_{\mathsf{A}}$-formula is real-world equivalent to an A-free formula.*

Proof. By Lemma 3.5, any $\mathcal{L}_{\mathsf{A}}$-formula ϕ is strictly equivalent to a truth-functional combination of A-atoms, that is of formulas of the forms p, $\mathsf{A}p$, $\Box\psi$ (with ψ being A-free), and $\mathsf{A}\Box\psi$ (again with ψ being A-free). But then ϕ is real-world equivalent to a truth-functional combination of formulas of the forms p and $\Box\psi$ (with ψ being A-free), because at the actual world of any Kripke model, $\mathsf{A}\theta$ has the same truth value as θ, for any $\mathcal{L}_{\mathsf{A}}$-formula θ. Truth-functional combinations of formulas p and $\Box\psi$ (with ψ being A-free) are, however, A-free. $\Box$

Extension of this result to logics with a non-normal semantics (i.e. whose models include so-called *non-normal worlds*), such as C.I. Lewis's $\mathsf{S2}$ and $\mathsf{S3}$, is straightforward: Lemma 3.2 holds in the weaker form that the formula $\Box(\mathsf{A}\phi \vee \psi)$ is strictly equivalent to $(\mathsf{A}\phi \wedge \Box\top) \vee \Box\psi$, and the rest of the argument is essentially unchanged. While we have explicitly treated only the case of mono-modal logics, it is clear that a similar construction can show the eliminability of the now-operator in propositional tense logics characterized by classes of tense-logical models. Extension of the eliminability result to broader classes of modal logics, with what are called *neighborhood semantics*, is addressed in §6.

4. Range of Logics Covered

In work on modal logics in $\mathcal{L}$, i.e. without an actuality operator, many authors part ways with Kripke [7, 8] in omitting mention of a designated actual world when defining a model to be a triple (W, R, V) consisting of a non-empty set W of worlds, an accessibility relation R over W, and a verification relation V. For want of a better term, we shall, following Chellas [2], call these *standard models* (as opposed to Kripke models, which were defined in §2). Standard models are naturally thought of as arising from a *relational frame* (W, R) by adding to it a verification relation V. There are then at least two natural ways of semantically characterizing systems of modal logic. First, given a class of relational frames, we may consider the set of all $\mathcal{L}$-formulas valid over all frames in the class (i.e. true at every world in every standard model based on a frame in the class); and second, given a class of standard models, we may consider the set of all $\mathcal{L}$-formulas true at every world in every standard model in the class. Clearly, the first construal is a special case of the second—given a class $\mathcal{C}$ of frames, simply consider the class of all standard models over frames in $\mathcal{C}$. At the same time, characterization of logics by means of classes of frames is both the more common and, generally, the more useful notion.

For one thing, it turns out that any system of modal logic characterized by a class of frames is automatically a *normal* modal logic, that is, axiomatizable by truth-functional tautologies, all instances of the distribution schema $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, some set of proper axiom schemata, and the rules of modus ponens and of necessitation (in general, the set of proper axiom schemata will not be recursive, though in all familiar cases it is finite). Many of the most familiar modal logics are normal.

So-called *incomplete* normal modal logics show that not every normal logic is characterized by a class of relational frames. However, even incomplete normal logics are characterized by a class of standard models: being normal, these logics possess canonical models, so that one characterizing class is always the singleton consisting of just the logic's canonical model. However, in the case of an incomplete normal logic characterized by a class of standard models $\mathcal{C}$, one can always find a pair of standard models based on the same relational frame of which one, but not the other, belongs to $\mathcal{C}$.

Let's say that a *Kripke frame* is a triple $(W, @, R)$, with (W, R) a relational frame in the sense defined above and $@$ a distinguished element of W. Adding a verification relation to a Kripke frame then results in a Kripke model as defined in §2. Obviously any standard model (W, R, V) can be transformed into a Kripke model by picking an element $w \in W$ and designating it as actual, resulting in (W, w, R, V).

Now suppose we have a system of modal logic characterized by a class $\mathcal{C}$ of standard models. Let $\mathcal{C}'$ be the class of all Kripke models (W, w, R, V) obtained from members (W, R, V) of $\mathcal{C}$ by designating a member w of W as actual. Then an $\mathcal{L}$-formula is true at the actual world of every Kripke model in $\mathcal{C}'$ just in case it is true at every world of every member of $\mathcal{C}$. Thus if we define the modal logic characterized by a class of Kripke models as the set of $\mathcal{L}$-formulas true at the actual world of every Kripke model in the class, it turns out that the logics characterized by $\mathcal{C}$ and by $\mathcal{C}'$ coincide. Similar remarks apply in the case of logics characterized by classes of relational vs. Kripke frames.

In designating one of the worlds in a model as the actual world and defining validity as truth at the actual world of every Kripke model, however, we gain a bit of flexibility.

The formula $\neg\Box\bot\wedge\neg\Box\neg\Box\bot$ cannot be a theorem of any consistent normal modal logic: application of necessitation to the left conjunct would yield a contradiction with the right (incidentally, the right conjunct implies the left, so the point could have been made with just $\neg\Box\neg\Box\bot$). Nevertheless, it is valid in any class of Kripke frames in which there is always at least one "terminal" world accessible from the actual world. Such (nonempty) classes of Kripke frames are thus distinct from any class $\mathcal{C}'$ of Kripke frames obtained from a class $\mathcal{C}$ of relational frames by letting $\mathcal{C}' = \{(W, w, R, V) \mid (W, R, V) \in \mathcal{C}, w \in W\}$. For further discussion of logics obtained by defining validity in such a way as to require truth only at designated worlds, see [9, ch. 3].

Let $\mathcal{C}$ be a class of Kripke models (which may, but need not, be obtained from a class of standard models in the way described above), and let S be the system of modal logic (in the language $\mathcal{L}$) characterized by it. By using the truth definition of §2, we can define the set $\mathsf{S}_{\mathsf{A}}(\mathcal{C})$ of all $\mathcal{L}_{\mathsf{A}}$-formulas true at the actual world of every member of $\mathcal{C}$, and we call $\mathsf{S}_{\mathsf{A}}(\mathcal{C})$ *the actuality extension of* S *with respect to* $\mathcal{C}$. Since adding the clause for A to the truth definition does not change the truth value of any $\mathcal{L}$-formula in any member of $\mathcal{C}$, we have the following:

Proposition 4.1. *If the logic* S *is characterized by the class* $\mathcal{C}$ *of Kripke models, the actuality extension of* S *with respect to* $\mathcal{C}$ *is a conservative extension of* S.

The same system of modal logic S may be characterized by two classes of models $\mathcal{C}_0$ and $\mathcal{C}_1$, but since the real-world equivalence of any $\mathcal{L}_{\mathsf{A}}$-formula with its A-free counterpart as per Theorem 3.6 holds across *all* Kripke models, it follows from Proposition 4.1 that, in such cases, $\mathsf{S}_{\mathsf{A}}(\mathcal{C}_0)$ and $\mathsf{S}_{\mathsf{A}}(\mathcal{C}_1)$ coincide. Systems of modal logic S characterized by classes of Kripke models thus have unique actuality extensions S_{A}.

Note that, since the proofs of §3 are formulated in terms of Kripke models rather than frames, the results concerning the eliminability of the actuality operator apply to the actuality extensions of *all* modal logics described in the preceding paragraphs.

5. Axiomatics for Actuality in Kripke Semantics

The result obtained in §3 and discussed in §4 is purely model-theoretic. This is in contrast to the work of Hazen [6], where a sound and complete natural-deduction style formalization of the actuality extension of $\mathsf{S5}$ with respect to the class of Kripke models with a universal accessibility relation was given, and it was shown how an arbitrary $\mathcal{L}_{\mathsf{A}}$-formula could be *proven* equivalent to an A-free formula. Crossley and Humberstone [3] focus on general rather than real-world validity but obtain a similar proof-theoretic result.

There is a "brute force" way of axiomatizing the actuality extension of any logic covered by the results of §3. Since an algorithm computing, for any A-containing formula ϕ, an A-free equivalent ψ can be extracted from the proof of Theorem 3.6, there is a recursive set of such conditionals $\psi \to \phi$. These conditionals can be added to S as axioms, with the proviso that only modus ponens (and no specifically modal rules like necessitation) can be applied to them or to formulas proved with their aid. Any valid A-containing formula ϕ of S_{A} will then have a proof two lines longer than a proof of its A-free equivalent ψ: first prove ψ, which is a theorem of S, then apply modus ponens to it and the conditional $\psi \to \phi$ that was added to axiomatize S_A. Inspection of the arguments in Section 3, however, reveals a less artificial way of axiomatizing the actuality extension of any *normal* modal logic S.

Begin by adding to an axiomatization of S all truth-functional tautologies of the language $\mathcal{L}_\mathsf{A}$ as well as all $\mathcal{L}_\mathsf{A}$-instances of the distribution schema as axioms (do not add $\mathcal{L}_\mathsf{A}$-instances of the proper axiom schemata of the original logic, though as we show below, that wouldn't hurt). Then add equivalential axiom schemata corresponding to the clauses of Lemma 3.1 and Lemma 3.2, as well as the rule of actualization: from ϕ infer $\mathsf{A}\phi$. Finally, add the axiom schema $\phi \leftrightarrow \mathsf{A}\phi$, but with the restriction that the rule of necessitation cannot be applied to its instances or to formulas derived with their aid.

Theorem 5.1. *Let $\mathcal{C}$ be a class of standard models characterizing the normal modal logic S. Then the axiomatization of S_A obtained from an axiomatization of S as described above is sound and complete with respect to the class $\mathcal{C}' = \{(W, w, R, V) | (W, R, V) \in \mathcal{C}, w \in W\}$ of Kripke models.*

Proof. For soundness, note that every theorem derived without appeal to the schema $\phi \leftrightarrow \mathsf{A}\phi$ is true at every world of every Kripke model in $\mathcal{C}'$. Every instance of $\phi \leftrightarrow \mathsf{A}\phi$ is true at the actual world of every Kripke model in $\mathcal{C}'$, and modus ponens preserves truth at the actual world. Hence any theorem of S_A is true at the actual world of every member of $\mathcal{C}'$.

For completeness, suppose that χ is true at the actual world of every member of $\mathcal{C}'$. Theorem 3.6 yields an A-free formula ψ that is real-world equivalent to χ. Hence ψ is also true at the actual world of every member of $\mathcal{C}'$. Being A-free, ψ is thus true at every world of every member of $\mathcal{C}$. Since S is characterized by $\mathcal{C}$, we know that ψ is a theorem of S and hence of its extension S_A. But S_A has sufficient resources to prove ψ equivalent to χ: Inspection of the proof of lemma 3.5 shows that truth-functional logic and the equivalences corresponding to lemmas 3.1 and 3.2, together with distribution and necessitation, suffice to prove χ equivalent to a truth-functional combination θ of A-atoms; truth-functional logic and the schema $\phi \leftrightarrow \mathsf{A}\phi$ then suffice to prove the equivalence of θ with ψ. □

As a special case, we obtain:

Corollary 5.2. *Let $\mathcal{C}$ be a class of relational frames characterizing the normal modal logic S. Then the axiomatization of S_A described above is sound and complete with respect to the class $\mathcal{C}' = \{(W, w, R) | (W, R) \in \mathcal{C}, w \in W\}$ of Kripke frames.*

It remains to discuss the question of inclusion, in the axiomatization of S_A, of $\mathcal{L}_\mathsf{A}$-instances of the axiom schemata proper to S. Theorem 5.1 shows that their inclusion is not necessary; we now make good on our announcement above that it wouldn't be detrimental either.

Proposition 5.3. *Let S be a normal modal logic. Any $\mathcal{L}_\mathsf{A}$-instance of an S-theorem (hence, in particular, any $\mathcal{L}_\mathsf{A}$-instance of a proper axiom schema of S) is provable in S_A without invoking any instances of the schema $\phi \leftrightarrow \mathsf{A}\phi$ (the* real-world schema *for short).*

Proof. Suppose $\mathsf{S} \vdash \phi$, and consider the substitution instance $\phi(\chi_1, \ldots, \chi_n)$ of ϕ, with each χ_i an $\mathcal{L}_\mathsf{A}$-formula. For the purposes of this proof, let S_A^- be S_A without the real-world schema. By mimicking the proof of Lemma 3.5, we can prove in $\mathsf{S}_{\mathsf{A}}^-$ that each χ_i is equivalent to a Boolean combination $\beta_i(\mathsf{A}\theta_1, \ldots, \mathsf{A}\theta_m)$ of A-atoms $\mathsf{A}\theta_1, \ldots, \mathsf{A}\theta_m$ such that A does not occur in β_i or any of the θ_j. To prevent notational clutter, we consider only the case $n = 1, m = 2$ for the remainder of the proof; however, the argument generalizes easily. Since we can prove in $\mathsf{S}_{\mathsf{A}}^-$ that χ_1 is equivalent to $\beta(\mathsf{A}\theta_1, \mathsf{A}\theta_2)$, we can also prove there that $\phi(\chi_1)$ is equivalent to $\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2))$. We claim that $\mathsf{S}_{\mathsf{A}}^-$ proves each of the following four conditionals:

(a) $(\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2) \to (\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2)) \leftrightarrow \phi(\beta_1(\top, \top)))$
(b) $(\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2) \to (\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2)) \leftrightarrow \phi(\beta_1(\top, \bot)))$
(c) $(\neg\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2) \to (\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2)) \leftrightarrow \phi(\beta_1(\bot, \top)))$
(d) $(\neg\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2) \to (\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2)) \leftrightarrow \phi(\beta_1(\bot, \bot)))$.

Assuming this for the time being, note that $\phi(\beta_1(\top, \top))$, $\phi(\beta_1(\top, \bot))$, $\phi(\beta_1(\bot, \top))$, and $\phi(\beta_1(\bot, \bot))$ are all $\mathcal{L}$-instances of the S-theorem ϕ, hence provable in the normal system S and hence in $\mathsf{S}_{\mathsf{A}}^-$. It follows that $\mathsf{S}_{\mathsf{A}}^-$ proves all of the following:

$$(\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2) \to \phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2))$$
$$(\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2) \to \phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2))$$
$$(\neg\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2) \to \phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2))$$
$$(\neg\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2) \to \phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2)).$$

But then, for truth-functional reasons, $\mathsf{S}_{\mathsf{A}}^-$ proves $\phi(\beta_1(\mathsf{A}\theta_1, \mathsf{A}\theta_2))$ and hence $\phi(\chi_1)$. It remains to show that $\mathsf{S}_{\mathsf{A}}^-$ indeed proves (a)–(d). For this it is sufficient to establish that $\mathsf{S}_{\mathsf{A}}^-$ proves (i) $\mathsf{A}\theta \to (\psi(\mathsf{A}\theta) \leftrightarrow \psi(\top))$ and (ii) $\neg\mathsf{A}\theta \to (\psi(\mathsf{A}\theta) \leftrightarrow \psi(\bot))$, for all $\mathcal{L}_{\mathsf{A}}$-formulas θ and ψ. We argue by induction on ψ and focus on (i). The interesting cases are where ψ is of the form $\#\eta$ with $\#$ being either A or $\Box$. By the induction hypothesis, we have $\mathsf{A}\theta \to (\eta(\mathsf{A}\theta) \leftrightarrow \eta(\top))$ in either case. The rule of actualization (respectively necessitation) and the axioms corresponding to Lemma 3.1 (respectively the distribution axioms) yield $\#\mathsf{A}\theta \to (\#\eta(\mathsf{A}\theta) \leftrightarrow \#\eta(\top))$ when $\#$ is A (respectively $\Box$). If $\# = \mathsf{A}$, we note that $\mathsf{A}\mathsf{A}\theta$ is equivalent to $\mathsf{A}\theta$ by the axioms corresponding to Lemma 3.1, and we're done. If $\# = \Box$, it suffices to show that $\mathsf{S}_{\mathsf{A}}^-$ proves $\mathsf{A}\theta \to \Box\mathsf{A}\theta$. This follows from the axioms corresponding to Lemmas 3.1 and 3.2: $(\mathsf{A}\neg\theta \vee \Box\mathsf{A}\theta) \leftrightarrow \Box(\mathsf{A}\neg\theta \vee \mathsf{A}\theta)$ is an instance of the latter. Its right hand side is easily seen to be provable in $\mathsf{S}_{\mathsf{A}}^-$, so we have the left hand side as a theorem, which is truth-functionally equivalent to $\mathsf{A}\theta \to \Box\mathsf{A}\theta$ (modulo the axioms corresponding to Lemma 3.1). □

Corollary 5.4. S_{A} *has the same set of theorems as the system that is just like* S_{A}*, except that it contains as additional axioms all* $\mathcal{L}_{\mathsf{A}}$*-instances of the proper axiom schemata of* S*.*

We note that inclusion of $\mathcal{L}_{\mathsf{A}}$-instances of the proper axiom schemata of S in the axiomatization of S_{A}, while not necessary, will permit simpler and (in the case of logics intended to capture aspects of the logic of natural language) more natural proofs.

6. Actuality in Neighborhood Semantics

An even broader class of modal logics can be defined, semantically, in terms of *neighborhood semantics.* A *neighborhood frame* is a triple $(W, @, N)$, where W and $@$ are as in the Kripke semantics described in §2, and N is a function mapping each

$w \in W$ to a set $N(w)$ of subsets of W. We refer to the elements of $N(w)$ as the *neighborhoods* of w. A *neighborhood model* is a quadruple $\mathbf{K} = (W, @, N, V)$, where $(W, @, N)$ is a neighborhood frame and V is a verification relation as in the Kripke semantics of §2.

Given a neighborhood model $\mathbf{K} = (W, @, N, V)$, truth at a world $w \in W$ is defined as in Kripke semantics for propositional variables and the truth-functional connectives. The truth conditions for necessitations are as follows: $\mathbf{K} \models_w \Box\phi$ if and only if $\{v \in W | \mathbf{K} \models_v \phi\} \in N(w)$, that is, the necessitation of ϕ is true at w if the set of all worlds at which ϕ is true is a neighborhood of w. As in the Kripke semantics, $\mathbf{K} \models_w \mathsf{A}\phi$ just in case $\mathbf{K} \models_@ \phi$, and we say that ϕ is true in $\mathbf{K}$, $\mathbf{K} \models \phi$, if $\mathbf{K} \models_@ \phi$. If $\mathcal{C}$ is a class of neighborhood models, ϕ is valid over $\mathcal{C}$ if $\mathbf{K} \models \phi$ for every $\mathbf{K} \in \mathcal{C}$.

Clearly the range of logics characterizable by classes of neighborhood models subsumes the range of logics characterizable by classes of Kripke models, since a Kripke model can be construed as a neighborhood model by putting into $N(w)$ just those subsets of W that include all worlds accessible from w. (Even non-normal worlds w, as occur in Kripke's [8] semantics for logics like $\mathsf{S2}$ and $\mathsf{S3}$, can be accommodated by letting $N(w)$ be empty for such w.)

For ease of exposition, when $(W, @, N)$ is a neighborhood frame, and $(W, @, N, V)$ a neighborhood model, let us call (W, N) a *minimal frame*, and (W, N, V) a *minimal model* (this conforms with the terminology in [2]). Thus neighborhood frames (neighborhood models) stand to minimal frames (minimal models) as Kripke frames (Kripke models) stand to relational frames (standard models). As is the case with relational semantics, many writers treating logics without actuality suppress mention of @ in the definitions of frames and models, that is, they work with minimal rather than neighborhood frames and models, and require truth at every world of every relevant model for validity. The theory of logics so characterizable is discussed in [2, ch. 7-9]. As in the case of Kripke models, designation of a world as actual allows more logics to be characterized: for example, we could require that W be a member of $N(@)$ without requiring it to be a member of every $N(w)$, and so make $\Box\top$ and $\top \leftrightarrow \Box\top$ valid without making $\Box\Box\top$ valid.

The actuality operator is eliminable in all modal logics characterized by classes of neighborhood models (i.e. the analog of Theorem 3.6 holds), but the proof given in §3 does not extend to them. While Lemma 3.1 holds without change, and Lemma 3.2 holds in the form that $\Box(\mathsf{A}\phi \vee \psi)$ is strictly equivalent to $(\mathsf{A}\phi \wedge \Box\top) \vee (\neg\mathsf{A}\phi \wedge \Box\psi)$, the proof of Lemma 3.5 breaks down because $\Box$ does not, in general, distribute over conjunction in these logics.

The basic idea for a proof of an analog of 3.6, however, can be seen by observing that, since $\mathsf{A}\phi$ is true at every world if ϕ is true at @ and at no world if ϕ is false at @, $\Box\mathsf{A}\phi$ will have the same truth value at every world as $\Box\top$ in models where ϕ is true at @, and the same truth value as $\Box\bot$ in models where ϕ is false at @. Generalizing the example, we have:

Lemma 6.1. *Let ϕ be an $\mathcal{L}_\mathsf{A}$-formula in which no occurrence of A is within the scope of an occurrence of $\Box$. Then for every neighborhood model $\mathbf{K}$ there is an A-free formula ψ which has the same truth value as $\Box\phi$ at every world of $\mathbf{K}$.*

Proof. Let $\mathsf{A}\theta_1, \ldots \mathsf{A}\theta_n$ be all the maximal subformulas of ϕ having an initial actuality operator. Then the desired ψ can be obtained from $\Box\phi$ by substituting $\top$ for each occurrence of an $\mathsf{A}\theta_i$ for which $\mathbf{K} \models_{@} \theta_i$ and $\bot$ for each occurrence of a θ_i for which $\mathbf{K} \not\models_{@} \theta_i$. $\square$

In the motivating example, we can eliminate the dependence on the model by writing $(\mathsf{A}\phi \wedge \Box\top) \vee (\neg\mathsf{A}\phi \wedge \Box\bot)$. Intuitively: The left conjuncts determine which disjunct to use in any given model, and then the right conjunct is the equivalent formula for that model. Generalizing, we have:

Lemma 6.2. *Let ϕ be an $\mathcal{L}_{\mathsf{A}}$-formula in which no occurrence of* A *is within the scope of an occurrence of $\Box$. Then there is a formula ψ which, for every neighborhood model $\mathbf{K}$, has the same truth value at every world of $\mathbf{K}$ as $\Box\phi$, and in which no occurrence of* A *is within the scope of an occurrence of $\Box$.*

Proof. Let $\mathsf{A}\theta_1, ...\mathsf{A}\theta_n$ be as before. Any given neighborhood model $\mathbf{K}$ assigns one of the two truth values to each $\mathsf{A}\eta_i$, and therefore any given $\mathbf{K}$ validates exactly one conjunction of the form $\sigma_1 \wedge \ldots \wedge \sigma_n$, where each σ_i is either $\mathsf{A}\theta_i$ or $\neg\mathsf{A}\theta_i$. Let $\eta_1, \ldots, \eta_m$ (where $m = 2^n$) be all such n- membered conjunctions with i-th conjunct either $\mathsf{A}\theta_i$ or $\neg\mathsf{A}\theta_i$. For each η_j, let χ_j be the result of the corresponding substitution in ϕ: that is, the result of substituting $\top$ for each occurrence of an $\mathsf{A}\theta_i$ which occurs as a conjunct in η_j, and substituting $\bot$ for each occurrence of an $\mathsf{A}\theta_i$ for which $\neg\mathsf{A}\theta_i$ occurs as a conjunct in η_j. Then the disjunction of the m conjunctions $\eta_j \wedge \Box\chi_j$ will be the desired ψ: in any model, exactly one of the η_j will be true (at all worlds), and the corresponding χ_j will, in that model, have the same truth value as ϕ at every world. $\square$

Lemma 6.2 shows that subformulas of the form $\mathsf{A}\theta$ occurring in the scope of a single $\Box$ can be pulled out of the necessitated formula. Induction on formulas then gives us:

Lemma 6.3. *For every $\mathcal{L}_{\mathsf{A}}$-formula ϕ there is an $\mathcal{L}_{\mathsf{A}}$-formula χ in which no occurrence of* A *is within the scope of an occurrence of $\Box$, such that ϕ and χ have the same truth value at every world of every neighborhood model.*

As an immediate consequence we have:

Theorem 6.4. *For every $\mathcal{L}_{\mathsf{A}}$-formula ϕ there is an* A*-free formula ψ such that ϕ and ψ have the same truth value at the actual world of every neighborhood model.*

7. Axiomatics for Actuality in Neighborhood Semantics

Let's say that a system S of modal logic in the language $\mathcal{L}$ is characterized by the class $\mathcal{C}$ of neighborhood models just in case S is the set of all $\mathcal{L}$-formulas that are true at the actual world of every member of $\mathcal{C}$. If S is so characterized by $\mathcal{C}$, we define $\mathsf{S}_{\mathsf{A}}(\mathcal{C})$ just as in the case of Kripke semantics as the set of all $\mathcal{L}_{\mathsf{A}}$-formulas that are true at the actual world of every member of $\mathcal{C}$. Just as before, it is easy to see from the eliminability result of the preceding section that, if S is characterized by $\mathcal{C}_0$ and by $\mathcal{C}_1$, $\mathsf{S}_{\mathsf{A}}(\mathcal{C}_0) = \mathsf{S}_{\mathsf{A}}(\mathcal{C}_1)$; in other words, the actuality extension of S is uniquely determined by S.

As in the case of logics with Kripke semantics, our model-theoretic eliminability result suggests a brute force axiomatization of the actuality extensions of such logics. Following a suggestion made in correspondence by Humberstone, however, we describe a more elegant axiomatization below, at least for systems of logic of the kind that Chellas [2] calls "classical". A *classical system of modal logic* in the language $\mathcal{L}$ is axiomatized by truth-functional tautologies, some proper axiom schemata, and the rules of modus ponens and $\Box$-congruence, i.e. the rule that licenses the inference from $\phi \leftrightarrow \psi$ to $\Box\phi \leftrightarrow \Box\psi$.

Suppose S is a classical system characterized by the class $\mathcal{C}$ of minimal models (there will always be such a class, since classical systems possess canonical models). Extend S by including all $\mathcal{L}_{\mathsf{A}}$-instances of truth-functional tautologies and all equivalential axioms as suggested by Lemma 3.1, by adding the rule of actualization, and by strengthening the rule of $\Box$-congruence to the rule ($\mathsf{A}\Box$-Cong) of A-conditional $\Box$-congruence (due to Humberstone): from $\mathsf{A}\theta \to (\phi \leftrightarrow \psi)$ infer $\mathsf{A}\theta \to (\Box\phi \leftrightarrow \Box\psi)$. Further add the schema $\phi \leftrightarrow \mathsf{A}\phi$, with the proviso that ($\mathsf{A}\Box$-Cong) must not be applied to its instances or to theorems derived with their aid.

We note that $\Box$-congruence is a derivable rule in the system thus axiomatized: Suppose $\phi \leftrightarrow \psi$ is provable. Then so is $\mathsf{A}\top \to (\phi \leftrightarrow \psi)$ for truth-functional reasons. Hence, by ($\mathsf{A}\Box$-Cong), $\mathsf{A}\top \to (\Box\phi \leftrightarrow \Box\psi)$ is a theorem. Since $\top$ is a tautology, it is provable, and by the rule of actualization, so is $\mathsf{A}\top$. But then modus ponens yields $\Box\phi \leftrightarrow \Box\psi$.

Letting $\mathcal{C}'$ be the class of neighborhood models $\{(W, w, N, V) | (W, N, V) \in \mathcal{C}, w \in W\}$, we claim that our axiomatization is sound and complete with respect to $\mathsf{S}_{\mathsf{A}} := \mathsf{S}_{\mathsf{A}}(\mathcal{C}')$.

For soundness, we first show that if the premise of an application of ($\mathsf{A}\Box$-Cong) is true at every world in every member of $\mathcal{C}'$, then so is its conclusion. So suppose that $\mathsf{A}\theta \to (\phi \leftrightarrow \psi)$ holds at every world of every member of $\mathcal{C}'$. Let $(W, @, N, V)$ be in $\mathcal{C}'$ and pick any $w \in W$ at which $\mathsf{A}\theta$ holds. Then $\mathsf{A}\theta$ holds at all worlds in W. Since $\mathsf{A}\theta \to (\phi \leftrightarrow \psi)$ also holds at all worlds in W, so does $\phi \leftrightarrow \psi$. But then ϕ and ψ hold at exactly the same worlds in $(W, @, N, V)$, and hence, at w, $\Box\phi \leftrightarrow \Box\psi$. It follows that the last line of any derivation not involving the schema $\phi \leftrightarrow \mathsf{A}\phi$ is true at all worlds in all members of $\mathcal{C}'$. Now $\phi \leftrightarrow \mathsf{A}\phi$ is true at the actual world of every neighborhood model, and modus ponens preserves real-world truth. Hence our axiomatization is sound.

For completeness, it suffices to check that the conversion implicit in the proof of Lemma 6.2 can be effected within our axiomatization. Recall that we start with a formula ϕ which does not contain A within the scope of an occurrence of $\Box$, and then show how to rewrite $\Box\phi$ in such a way that A still doesn't occur within the scope of an occurrence of $\Box$. The rewriting process begins by identifying the maximal subformulas of ϕ that begin with A, say $\mathsf{A}\theta_1, \ldots, \mathsf{A}\theta_n$. Again, to reduce notational clutter, we focus on the case $n = 2$. We write ϕ as $\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2)$. Since A does not occur within the scope of a $\Box$ in ϕ, we can prove each of the following four conditionals:

(a) $\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \to (\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \phi(\top, \top))$
(b) $\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \to (\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \phi(\top, \bot))$
(c) $\neg\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \to (\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \phi(\bot, \top))$
(d) $\neg\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \to (\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \phi(\bot, \bot))$.

By the equivalences of Lemma 3.1, the antecedents of (a)–(d) are equivalent to $\mathsf{A}(\theta_1 \wedge \theta_2)$, $\mathsf{A}(\theta_1 \wedge \neg\theta_2)$, $\mathsf{A}(\neg\theta_1 \wedge \theta_2)$, and $\mathsf{A}(\neg\theta_1 \wedge \neg\theta_2)$, respectively, so that application of ($\mathsf{A}\Box$-Cong) to (a)–(d) yields

(a') $\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \to (\Box\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \Box\phi(\top, \top))$
(b') $\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \to (\Box\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \Box\phi(\top, \bot))$
(c') $\neg\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \to (\Box\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \Box\phi(\bot, \top))$

(d') $\neg\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \to (\Box\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2) \leftrightarrow \Box\phi(\bot, \bot))$.

For truth-functional reasons we then have $\Box\phi(\mathsf{A}\theta_1, \mathsf{A}\theta_2)$ equivalent to the disjunction of $\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \wedge \Box\phi(\top, \top)$, $\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \wedge \Box\phi(\top, \bot)$, $\neg\mathsf{A}\theta_1 \wedge \mathsf{A}\theta_2 \wedge \Box\phi(\bot, \top)$, and $\neg\mathsf{A}\theta_1 \wedge \neg\mathsf{A}\theta_2 \wedge \Box\phi(\bot, \bot)$, which is the desired formula. Our axiomatization of S_A is therefore complete.

As in the case of Kripke semantics, we are free to add, to the axioms for S_A, all $\mathcal{L}_\mathsf{A}$-instances of the proper axiom schemata of S, which are provable from our axiomatization without invoking the real-world schema, but might reasonably be regarded as axioms.

Appendix: An Application

Williamson [12] considers T_A, the actuality extension of the basic normal alethic logic T, and proves that in it the compound operator $\mathsf{A}\Box$ behaves like $\Box$ in $\mathsf{S5}$: more precisely, a formula in which the A and $\Box$ operators occur only as parts of the compound $\mathsf{A}\Box$ (call these *Williamson-formulas* or *W-formulas* for short) is valid in T_A if and only if the formula obtained from it by deleting all occurrences of A is valid in $\mathsf{S5}$. His proof naturally generalizes from T to any normal system between T and $\mathsf{S5}$. Our eliminability results provide an alternative proof of this fact.

The conversion algorithm implicit in the proof of Lemma 3.5 will transform a W-formula ϕ into a strictly equivalent W-formula χ which is first degree in the compound operator, i.e. no $\mathsf{A}\Box$ occurs inside the scope of another. In other words, every W-formula ϕ is strictly equivalent to a Boolean combination χ of propositional variables and formulas of the form $\mathsf{A}\Box\theta$, with θ containing neither A nor $\Box$. We note that this conversion invokes, besides truth-functional logic and the inference from the strict equivalence of formulas σ and ρ to the strict equivalence of $\mathsf{A}\Box\sigma$ and $\mathsf{A}\Box\rho$, only distribution of $\mathsf{A}\Box$ over conjunctions, the equivalencies mentioned in Lemma 3.1, and the consequence of Lemmas 3.1 and 3.2 according to which $\mathsf{A}\Box(\mathsf{A}\Box\sigma \vee \rho)$ is strictly equivalent to $\mathsf{A}\Box\sigma \vee \mathsf{A}\Box\rho$.

By Theorem 3.6, χ is real-world equivalent to the result ψ of deleting its occurrences of A, which is a first degree modal formula in the ordinary operators.

Let S be any normal modal logic between T and $\mathsf{S5}$. Then ϕ is valid in S_A iff χ is valid in S_A, iff ψ is valid in S_A, iff ψ is valid in S (by conservativity of S_A over S, Proposition 4.1), iff ψ is valid in $\mathsf{S5}$. The last equivalence follows from the familiar fact that all the alethic modal logics from T to $\mathsf{S5}$ (inclusive) have the same set of valid first degree formulas.

It remains to show that, in $\mathsf{S5}$, one can prove that ψ is equivalent to the result of erasing all occurrences of A in ϕ. But this can be done, essentially by reversing the transformation of ϕ into χ but with A erased. This is possible within $\mathsf{S5}$ because all the resources used in the conversion are available in $\mathsf{S5}$ when A is erased: We have truth-functional logic, the distribution axioms and necessitation, hence also the inference from $\sigma \leftrightarrow \rho$ to $\Box\sigma \leftrightarrow \Box\rho$; $\Box$ distributes over conjunctions; and the equivalencies of Lemma 3.1 become trivial when A is erased. Consider finally equivalence of $\mathsf{A}\Box(\mathsf{A}\Box\sigma \vee \rho)$ with $\mathsf{A}\Box\sigma \vee \mathsf{A}\Box\rho$. Upon deletion of A, this becomes the equivalence of $\Box(\Box\sigma \vee \rho)$ with $\Box\sigma \vee \Box\rho$, which is valid in $\mathsf{S5}$.

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