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Fragments of *HA* based on Σ_1-induction

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Abstract. In the first part of this paper we investigate the intuitionistic version $iI\Sigma_1$ of $I\Sigma_1$ (in the language of *PRA*), using Kleene's recursive realizability techniques. Our treatment closely parallels the usual one for *HA* and establishes a number of nice properties for $iI\Sigma_1$, e.g. existence of primitive recursive choice functions (this is established by different means also in [D94]). We then sharpen an unpublished theorem of Visser's to the effect that quantifier alternation alone is much less powerful intuitionistically than classically: $iI\Sigma_1$ together with induction over arbitrary prenex formulas is Π_2-conservative over $iI\Pi_2$.

In the second part of the article we study the relation of $iI\Sigma_1$ to $iI\Pi_1$ (in the usual arithmetical language). The situation here is markedly different from the classical case in that $iI\Pi_1$ and $iI\Sigma_1$ are mutually incomparable, while $iI\Sigma_1$ is significantly stronger than $iI\Pi_1$ as far as provably recursive functions are concerned: All primitive recursive functions can be proved total in $iI\Sigma_1$ whereas the provably recursive functions of $iI\Pi_1$ are all majorized by polynomials over $\mathbb{N}$. $iI\Pi_1$ is unusual also in that it lacks closure under Markov's Rule MR_{PR}.

1 Preliminaries

$L(PRA)$ is the first-order language containing function symbols for each primitive recursive function term. *iPRA* is the intuitionistic theory in the language $L(PRA)$ whose nonlogical axioms are the defining equations for all primitive recursive functions plus the axiom scheme of induction restricted to atomic formulas. Since in *iPRA* every Δ_0-formula A (i.e. every quantifier occurring in A is bounded) of $L(PRA)$ is equivalent to an equation $t = 0$, induction over Δ_0-formulas is provable in *iPRA*. *PRA* is *iPRA* augmented by classical logic. $iI\Sigma_1^+$ is *iPRA* plus induction

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over formulas of the form $\exists y \varphi(x, y, \bar{z})$ with φ atomic, and $I\Sigma_1^+$ is $iI\Sigma_1^+$ together with classical logic.

L is the usual arithmetical language given by $0, 1, +, \cdot$ and $<$. $iI\Delta_0$ is the intuitionistic L-theory axiomatized by the usual axioms for PA^- (cf. e.g. [K91]; the obvious bound should be put on the only existential quantifier) plus induction over Δ_0-formulas of L. More generally, if Γ is any class of L-formulas, the theory $iI\Gamma$ is $iI\Delta_0$ together with the axiom schema of induction restricted to formulas from Γ. Note that by arguments as in [S73], each fragment $iI\Gamma$ of HA has the disjunction property (DP) and the explicit definability property (ED). $I\Gamma$ is $iI\Gamma$ with classical logic.

Σ_1 is the class of L-formulas of the form $\exists \bar{y} \varphi(x, \bar{y}, \bar{z})$, Π_1 the class of L-formulas $\forall \bar{y} \varphi(x, \bar{y}, \bar{z})$ with, in both cases, φ in Δ_0. Similarly for the other formula classes in the arithmetical hierarchy. Analogous definitions apply to formulas of $L(PRA)$, where in the presence of coding functions we may assume that no two quantifiers of the same kind appear consecutively in the prefix and that the matrix is atomic. Given a formula class Γ, $\neg\Gamma$ is the class of formulas of the form $\neg\varphi$ with $\varphi \in \Gamma$.

We collect a number of facts about the theories introduced above.

Fact 1. In each of the theories defined above, atomic and Δ_0-formulas are decidable.

Fact 2 (Parsons). $I\Sigma_1^+$ is Π_2-conservative over *PRA*.

In fact, whenever $I\Sigma_1^+ \vdash \forall x \exists y \varphi(x, y)$ where $\varphi(x, y)$ is quantifier-free and contains at most x, y free, then for some primitive recursive function term f, $PRA \vdash \forall x \varphi(x, fx)$.

For an accessible proof of Parsons' theorem, cf. [P92] (where, for reasons of convenience, reference is made to truth in the standard model instead of provability in *PRA*).

Definition 1. *We define an operation $\phi \mapsto \phi^-$ on formulas by induction on ϕ:*

- *If ϕ is atomic, $\phi^- \equiv \neg\neg\phi$.*
- $(\psi \vee \chi)^- \equiv \neg(\neg\psi^- \wedge \neg\chi^-)$.
- $(\psi \wedge \chi)^- \equiv \psi^- \wedge \chi^-$.
- $(\psi \to \chi)^- \equiv \psi^- \to \chi^-$.
- $(\exists x \psi)^- \equiv \neg\forall x \neg\psi^-$.
- $(\forall x \psi)^- \equiv \forall x \psi^-$.

The operation just defined is called the *negative translation*. The negative translation can be viewed as an embedding of classical into intuitionistic logic, due to the fact that the provability of ϕ in classical predicate logic entails the provability of ϕ^- in intuitionistic logic, which is easily proved by induction on the derivation of ϕ in a suitable calculus, cf. [TVD88]. In fact, this property of the negative translation extends to *PA* and *HA*, since negative translations of induction axioms are again induction axioms. Note that in the context of theories with decidable atomic formulas, there is no need to double-negate atomic formulas in constructing ϕ^-. The negative translation also works in the following cases:

Fact 3. Let A be a formula and A^- its negative translation. Then we have:

1. $PRA \vdash A \Rightarrow iPRA \vdash A^-$;
2. $I\Delta_0 \vdash A \Rightarrow iI\Delta_0 \vdash A^-$.

Proof. This follows immediately from the corresponding result for pure logic (cf. e.g. [TVD88]) by noting that negative translations of quantifier-free and Δ_0-formulas are again quantifier-free and Δ_0, respectively.

Definition 2. *Given formulas ϕ and ρ, the Friedman translation of ϕ by ρ, denoted ϕ^ρ, is obtained from ϕ by replacing each atomic subformula P in ϕ by $P \vee \rho$ (where it is understood that no variable occurring free in ρ is bound in ϕ).*

Fact 4. *iPRA* and $iI\Delta_0$ are closed under the Friedman translation, i.e. if T is *iPRA* or $iI\Delta_0$ and $T \vdash \phi$, then also $T \vdash \phi^\rho$ for any formula ρ of the appropriate language.

Proof. One can show by induction on the formula ψ that $iI\Delta_0 \vdash \psi^\rho \leftrightarrow (\psi \vee \rho)$ for $\psi \in \Delta_0$ and arbitrary ρ, and similarly for *iPRA*. Now consider any induction axiom of $iI\Delta_0$, say $\psi(0) \wedge \forall x(\psi(x) \to \psi(Sx)) \to \psi(z)$ with $\psi \in \Delta_0$. Its Friedman translation by ρ is then equivalent to $(\psi(0) \vee \rho) \wedge \forall x((\psi(x) \vee \rho) \to (\psi(Sx) \vee \rho)) \to (\psi(z) \vee \rho)$. Argue in $iI\Delta_0$. Assume the antecedent and let z be arbitrary. We must show that $\psi(z) \vee \rho$. Note that Δ_0-formulas are decidable in $iI\Delta_0$. If $\psi(z)$ holds, we have nothing to show. Otherwise, by $\forall x(\psi(x) \vee \neg\psi(x))$, $\neg\psi(z)$. Now the least number principle for Δ_0-formulas is provable in $iI\Delta_0$ (due to decidability of Δ_0-formulas); so we may assume that z is minimal with the property $\neg\psi(z)$. Then either $z = 0$, in which case by $\psi(0) \vee \rho$ we obtain ρ, or $z = Sx$ for some x. By minimality of z we have $\psi(x)$ and hence, by $\forall x((\psi(x) \vee \rho) \to (\psi(Sx) \vee \rho))$ also $\psi(z) \vee \rho$. By $\neg\psi(z)$ we again obtain ρ, q.e.d. The proof for *iPRA* is similar but simpler.

Remark. There is also a model-theoretic proof of fact 4. Observe that the Kripke models of *iPRA* are precisely those Kripke structures that have *classical* models of *PRA* attached to each node. Then apply the 'First Pruning Lemma' of [VMKV86]. To apply the same proof to the case of $iI\Delta_0$, note in addition that the Kripke models of $iI\Delta_0$ are precisely those Kripke structures having classical models of $I\Delta_0$ at each node such that, whenever node α is below node β, the structure attached to α is a Δ_0-elementary substructure of the structure attached to β.

Fact 5. *PRA* is Π_2-conservative over *iPRA*, $I\Delta_0$ is Π_2-conservative over $iI\Delta_0$.

Proof. Suppose $PRA \vdash \exists y \varphi(x, y)$ with φ atomic. By fact 3, $iPRA \vdash \neg\neg\exists y \varphi(x, y)$. Letting ρ be the formula $\exists z \varphi(x, z)$, we have by fact 4 that $iPRA \vdash (\neg\neg\exists y \varphi(x, y))^\rho$, and this formula is $(\exists y(\varphi(x, y) \vee \exists z \varphi(x, z)) \to \exists y \varphi(x, y)) \to \exists y \varphi(x, y)$. It is easy to see that the premise of this implication is derivable, hence $iPRA \vdash \exists y \varphi(x, y)$. Similarly for $iI\Delta_0$.

Fact 6. $I\Sigma_1^+$ is Π_2-conservative over $iI\Sigma_1^+$.

Proof. This is immediate from facts 2 and 5.

Fact 7. $iI\Sigma_1^+$ is conservative over (in fact, a definitional extension of) $iI\Sigma_1$.

Proof. The primitive recursive functions can be defined in $iI\Sigma_1$ in the usual way (e.g. using Gödel's β-function).

Fact 8. $I\Sigma_1$ is Π_2-conservative over $iI\Sigma_1$.

Proof. By facts 6 and 7.

Fact 9. $iI\Sigma_1^+$ and $iI\Sigma_1$ are closed under MR_{PR}, i.e. if T is one of $iI\Sigma_1^+$ and $iI\Sigma_1$ and $T \vdash \neg\neg\exists\overline{y}\varphi(\overline{x},\overline{y})$ with $\varphi \in \Delta_0$, then also $T \vdash \exists\overline{y}\varphi(\overline{x},\overline{y})$.

Proof. If $iI\Sigma_1 \vdash \neg\neg\exists\overline{y}\varphi(\overline{x},\overline{y})$, then $I\Sigma_1 \vdash \exists\overline{y}\varphi(\overline{x},\overline{y})$. If φ is Δ_0, by fact 8 $iI\Sigma_1 \vdash \exists\overline{y}\varphi(\overline{x},\overline{y})$. Similarly for $iI\Sigma_1^+$, using fact 6.

2 $iI\Sigma_1^+$ and recursive realizability

We will now investigate the theory $iI\Sigma_1^+$ using Kleene's notion of recursive realizability. Our treatment parallels the one given for *HA* by Dragalin in [Dr87]. A comprehensive treatment of realizability can be found in [T73] and [T92]. We begin by recalling some standard concepts.

Definition 3. *For every formula ϕ of $iI\Sigma_1^+$ and variable x not occurring free in ϕ we define new formulas $x\,\mathbf{r}\phi$ (read: x recursively realizes ϕ) and $x\,\mathbf{q}\phi$ (read: x q-realizes ϕ) by induction on the formation of ϕ. $\mathbf{s}$ always denotes one of $\mathbf{r}$ and $\mathbf{q}$. Here, T denotes Kleene's T-predicate (we suppress mention of arities) and U is the result-extracting function of Kleene's normal form.*

1. $x\,\mathbf{s}\bot :\equiv \bot$
2. $x\,\mathbf{s}t_0 = t_1 :\equiv t_0 = t_1$
3. $x\,\mathbf{s}(\varphi\&\psi) :\equiv ((x)_0\mathbf{s}\varphi)\&((x)_1\mathbf{s}\psi)$
4. $x\,\mathbf{s}(\varphi \vee \psi) :\equiv ((x)_0 = 0 \rightarrow (x)_1\mathbf{s}\varphi)\&((x)_0 \neq 0 \rightarrow (x)_1\mathbf{s}\psi)$
5. $x\,\mathbf{r}(\varphi \rightarrow \psi) :\equiv \forall y(y\,\mathbf{r}\varphi \rightarrow \exists zT(x,y,z))\&\forall yz((y\,\mathbf{r}\varphi)\&T(x,y,z) \rightarrow Uz\,\mathbf{r}\psi)$; $x\,\mathbf{q}(\varphi \rightarrow \psi) :\equiv \forall y(y\,\mathbf{q}\varphi \rightarrow \exists zT(x,y,z))\&\forall yz((y\,\mathbf{q}\varphi)\&T(x,y,z) \rightarrow Uz\,\mathbf{q}\psi)\&(\varphi \rightarrow \psi)$
6. $x\,\mathbf{s}\forall y\psi(y) :\equiv \forall y\exists zT(x,y,z)\&\forall yz(T(x,y,z) \rightarrow Uz\,\mathbf{s}\psi(y))$
7. $x\,\mathbf{s}\exists y\psi(y) :\equiv (x)_0\mathbf{s}\psi((x)_1)$

Definition 4. *The partial recursive (p.r.) terms are defined inductively by the following clauses:*

1. 0 *and all individual variables are p.r. terms.*
2. *If f is an n-ary primitive recursive function symbol and $t_1,...,t_n$ are p.r. terms, then $ft_1...t_n$ is a p.r. term.*
3. *If $t,t_1,...,t_n$ are p.r. terms, then so is $\{t\}(t_1,...,t_n)$.*

Definition 5. *For any p.r. term t with $FV(t) = \overline{x}$ and every variable $y \notin \overline{x}$ we define, by induction on the generation of t, a formula $eq(t,y)$ of $iI\Sigma_1^+$ such that $FV(eq(t,y)) = \overline{x}, y$. (We later write and interpret $eq(t,y)$ as $t = y$.)*

1. *If t is 0 or an individual variable other than y, we put $eq(t,y) :\equiv t = y$.*
2. *$eq(ft_1...t_n, y) :\equiv \exists\overline{z}(\bigwedge eq(t_i, z_i) \& f\overline{z} = y)$.*
3. *$eq(\{t\}(t_1, ..., t_n), y) :\equiv \exists z \exists \overline{w}(eq(t,z) \& \bigwedge eq(t_i, w_i) \& eq(\{z\}(\overline{w}), y))$, where for variables $z, \overline{w}$ we put:*
   $eq(\{z\}(\overline{w}), y) :\equiv \exists x(T_n(z, \overline{w}, x) \& Ux = y)$.

Definition 6. *For p.r. terms t, r put*

- $t! :\equiv \exists y\, t = y\, (:\equiv \exists y\, eq(t,y))$
- $t = r :\equiv \exists y\, (t = y \& r = y)\ (:\equiv \exists y\, (eq(t,y) \& eq(r,y)))$
- $t \simeq r :\equiv \forall y(t = y \leftrightarrow r = y)(:\equiv \forall y(eq(t,y) \leftrightarrow eq(r,y)))$

We now list a number of lemmas concerning some facts of elementary recursion theory which can be proved in $iI\Sigma_1^+$ by formalizing the usual proofs.

Lemma 1. *For each p.r. term t with $FV(t) \subseteq \overline{x}$ there is an $n \in \mathbb{N}$ with $iI\Sigma_1^+ \vdash t \simeq \{n\}(\overline{x})$.*

Lemma 2. *If t is a closed p.r. term with value $n \in \mathbb{N}$, then $iI\Sigma_1^+ \vdash t = n$.*

Lemma 3 (S_n^1-theorem). *For each p.r. term t, $\{t\}(x, \overline{y}) \simeq \{S_n^1(t, \overline{y})\}(x)$ is provable in $iI\Sigma_1^+$.*

Lemma 4. *For each p.r.term t and variable x there is a term t_1 of the language of $iI\Sigma_1^+$, $FV(t_1) = FV(t) - \{x\}$, such that $iI\Sigma_1^+ \vdash \{t_1\}(x) \simeq t$. The term t_1 will be denoted $\Lambda x.t$.*

Lemma 5 (Recursion theorem). *For each p.r. term $t(x, \overline{y})$ with variables as shown there is an $e \in \mathbb{N}$ such that $iI\Sigma_1^+ \vdash \{e\}(\overline{y}) \simeq t(e, \overline{y})$.*

We are now ready to prove soundness for the notions of recursive and q-realizability. Again, we follow the treatment of [Dr87].

Theorem 1. *If $iI\Sigma_1^+ \vdash \varphi$, there is a p.r. term t in at most the parameters of φ such that $iI\Sigma_1^+ \vdash \exists y\, (t = y \& y\, \mathbf{s}\, \varphi)$, where* **s** *is either of* **r** *and* **q***.*

Proof. We proceed by induction on the $iI\Sigma_1^+$-derivation of φ. For definiteness the reader may assume the Hilbert-type formalization *HPC* of intuitionistic predicate logic as in [Dr87]. Except for the realizability of the induction axioms, everything works exactly as for *HA*, because induction is used in the formal proof of realizability of the induction axioms only. We will thus concentrate on the Σ_1-induction axioms. By Lemma 1, there are $n, e, f \in \mathbb{N}$ such that (we now write $\vdash$ for $iI\Sigma_1^+ \vdash$)

- $\vdash \{n\}(v, x, y) \simeq \{\{(v)_1\}(x)\}(y)$
- $\vdash \{e\}(z, v, x) \simeq (v)_0$
- $\vdash \{f\}(z, v, x) \simeq \{n\}(v, pred(x), \{z\}(v, pred(x)))$.

By the formalized recursion theorem there is a natural number $m \in \mathbb{N}$ such that

- $\vdash \{m\}(v, 0) \simeq (v)_0$ and
- $\vdash \{m\}(v, x+1) \simeq \{n\}(v, x, \{m\}(v, x))$

(Take e.g. $m \in \mathbb{N}$ with $\vdash \{m\}(v, x) \simeq \{\overline{sg}(x) \cdot e + sg(x) \cdot f\}(m, v, x)$.) We claim that $\Lambda v.(\Lambda x.\{m\}(v, x))$ provably realizes each induction axiom of $iI\Sigma_1^+$. Note that since $t :\equiv \Lambda v.(\Lambda x.\{m\}(v, x))$ is a term of the language of $iI\Sigma_1^+$, it is clear that $iI\Sigma_1^+ \vdash \exists y(t = y)$. Consider the instance

$$\phi :\equiv \varphi(0) \& \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x),$$

where $\varphi(x) \equiv \exists y \psi(x, y, \overline{z})$ and ψ is atomic. Argue in $iI\Sigma_1^+$. We need to show that $t\, \mathbf{r}\, \phi$, i.e.

1. $\forall v(v\, \mathbf{r}(\varphi(0) \& \forall x(\varphi(x) \to \varphi(x+1))) \to \exists z T(t, v, z))$ and
2. $\forall vz(v\, \mathbf{r}(\varphi(0) \& \forall x(\varphi(x) \to \varphi(x+1))) \& T(t, v, z) \to Uz\, \mathbf{r} \forall x \varphi(x))$.

Assume $v\, \mathbf{r}(\varphi(0) \& \forall x(\varphi(x) \to \varphi(x+1)))$, i.e.

$$(v)_0\, \mathbf{r}\, \varphi(0) \text{ and } (v)_1\, \mathbf{r} \forall x(\varphi(x) \to \varphi(x+1)),$$

the latter being $\forall x \exists w T((v)_1, x, w) \& \forall x \forall w(T((v)_1, x, w) \to U w\, \mathbf{r}(\varphi(x) \to \varphi(x+1)))$. By definition of Λ we have $\vdash \{t\}(v) \simeq \Lambda x.\{m\}(v, x)$. The right hand side is a term of $iI\Sigma_1^+$, so $\vdash \exists y\{t\}(v) = y$. Since t itself is also an $iI\Sigma_1^+$-term, this is essentially $\exists z T(t, v, z)$, i.e. 1. It remains to show 2. So assume in addition that $T(t, v, z)$. We have to prove $Uz\, \mathbf{r} \forall x \varphi(x)$. Now $Uz \simeq \{t\}(v) \simeq \Lambda x.\{m\}(v, x)$, so we must establish that $\Lambda x.\{m\}(v, x)\, \mathbf{r} \forall x \varphi(x)$, i.e.

$$\forall x \exists y T(\Lambda x.\{m\}(v, x), x, y)$$

and

$$\forall x \forall y(T(\Lambda x.\{m\}(v, x), x, y) \to Uy\, \mathbf{r}\, \varphi(x)).$$

Modulo elementary properties of the T-predicate, these conditions are equivalent to

$$(\dagger) \quad \forall x \exists y(T(\Lambda x.\{m\}(v, x), x, y) \& Uy\, \mathbf{r}\, \varphi(x)).$$

To prove this formula by Σ_1-induction, we must first compute the complexity of $Uy\, \mathbf{r}\, \varphi(x)$. But

$$Uy\, \mathbf{r}\, \varphi(x) \equiv Uy\, \mathbf{r} \exists y \psi(x, y, \overline{z}) \equiv (Uy)_0 \mathbf{r}\, \psi(x, (Uy)_1, \overline{z}) \equiv \psi(x, (Uy)_1, \overline{z})$$

since ψ is atomic.

So we may induct on the formula $\exists y(T(\Lambda x.\{m\}(v, x), x, y) \& Uy\, \mathbf{r}\, \varphi(x))$. If $x = 0$, we know that $\{\Lambda x.\{m\}(v, x)\}(0) \simeq \{m\}(v, 0) \simeq (v)_0$, and by assumption $(v)_0\, \mathbf{r}\, \varphi(0)$. Induction step: We have $\{\Lambda x.\{m\}(v, x)\}(x+1) \simeq \{m\}(v, x+1) \simeq \{n\}(v, x, \{m\}(v, x)) \simeq \{\{(v)_1\}(x)\}(\{m\}(v, x))$. By the induction hypothesis, $\{m\}(v, x)$ is defined and realizes $\varphi(x)$. By $(v)_1\, \mathbf{r} \forall x(\varphi(x) \to \varphi(x+1))$, it then follows that $\{\{(v)_1\}(x)\}(\{m\}(v, x))$ is defined and realizes $\varphi(x+1)$, which is what we need.

Note that, as usual, the same realizing terms work for **q**-realizability.

For later use we mention the following corollary to the proof of Theorem 1:

Lemma 6. *If $i I\Sigma_3^+ \vdash \varphi$, there is a p.r. term t in at most the parameters of φ such that $i I\Pi_2^+ \vdash \exists y (t = y \,\&\, y\, \mathbf{r}\, \varphi)$.*

Proof. Redo the proof of Theorem 1. We must show by Π_2-induction the conjunction of the two formulas preceding (†) in the proof of Theorem 1, where φ is now Σ_3, say $\varphi(x) \equiv \exists a \forall b \exists c \psi(a, b, c, x)$ with ψ atomic. Again, we have to check the complexity of $Uy\, \mathbf{r}\, \varphi(x)$. Now

$$\begin{aligned}
&e\, \mathbf{r}\, \exists a \forall b \exists c \psi(a, b, c, x)\\
&\equiv (e)_0\, \mathbf{r}\, \forall b \exists c \psi((e)_1, b, c, x)\\
&\equiv \forall b \exists d (T((e)_0, b, d) \,\&\, Ud\, \mathbf{r}\, \exists c \psi((e)_1, b, c, x))\\
&\equiv \forall b \exists d (T((e)_0, b, d) \,\&\, (Ud)_0\, \mathbf{r}\, \psi((e)_1, b, (Ud)_1, x))\\
&\equiv \forall b \exists d (T((e)_0, b, d) \,\&\, \psi((e)_1, b, (Ud)_1, x)),
\end{aligned}$$

and this last formula is indeed Π_2.

From the soundness theorem we can now derive the usual corollaries. Of particular interest is Corollary 1 below, where we show that the definable (i.e. provably total) functions of $i I\Sigma_1^+$ are precisely the primitive recursive functions. A related but weaker result is obtained by Damnjanovic in [D94] using his more elaborate method of *strictly primitive recursive realizability*. Note that this result corresponds nicely to the case of *HA*, where an easy realizability argument shows that the definable functions are all provably recursive in *HA* (and thus in *PA*): If $HA \vdash \forall x \exists y \varphi(x, y)$, where φ is arbitrary and has only x, y free, then for some $e \in \mathbb{N}$, $HA \vdash e\, \mathbf{q}\, \forall x \exists y \varphi(x, y)$, so $HA \vdash \forall x \exists z T(e, x, z) \& \forall x \forall z (T(e, x, z) \rightarrow (Uz)_0\, \mathbf{q}\, \varphi(x, (Uz)_1))$. In particular, $HA \vdash \forall x \exists z T(e, x, z)$, so e is the index of a provably recursive function of *HA*. It follows (using the usual property of $\mathbf{q}$) that $\mathbb{N} \models \forall x \varphi(x, (f(x))_1)$ where f is the $(< \varepsilon_0)$-recursive function with index e. So the provably recursive functions coincide with the definable functions in the cases of both $i I\Sigma_1^+$ (proved below) and *HA*.

Corollary 1.

1. *$i I\Sigma_1^+$ has primitive recursive choice functions, i.e. whenever $i I\Sigma_1^+ \vdash \forall x \exists y \varphi (x, y)$, where $\varphi(x, y)$ is an arbitrary formula having only x, y free, then there is a primitive recursive function symbol f such that*

$$i I\Sigma_1^+ \vdash \forall x \varphi(x, fx).$$

2. *Let $\forall x_1 \exists y_1 ... \forall x_n \exists y_n \varphi(x_1, y_1, ..., x_n, y_n)$ be a prenex formula with no two consecutive quantifiers of the same kind in the prefix and a quantifier-free matrix φ. If $i I\Sigma_1^+ \vdash \forall x_1 \exists y_1 ... \forall x_n \exists y_n \varphi(x_1, y_1, ..., x_n, y_n)$, then there are primitive recursive function symbols $f_1, ... f_n$ such that*

$$i I\Sigma_1^+ \vdash \forall x_1 ... x_n \varphi(x_1, f_1(x_1), x_2, f_2(x_1, x_2), ..., x_n, f_n(x_1, ... x_n)).$$

Proof. Suppose $i I\Sigma_1^+ \vdash \forall x \exists y \varphi(x, y)$. By the soundness theorem, there is a closed p.r. term t with $i I\Sigma_1^+ \vdash \exists y (t = y \,\&\, y\, \mathbf{q} \forall x \exists y \varphi(x, y))$. Now t is closed and proven in $i I\Sigma_1^+$ to exist, so it has a value $e \in \mathbb{N}$, and by lemma 2,

$iI\Sigma_1^+ \vdash t = e$. This yields $iI\Sigma_1^+ \vdash e\,\mathbf{q}\forall x\exists y\varphi(x,y)$. $e\,\mathbf{q}\forall x\exists y\varphi(x,y)$ is the sentence $\forall x\exists yT(e,x,y)\&\forall xy(T(e,x,y) \to Uy\,\mathbf{q}\exists y\varphi(x,y))$. In particular $iI\Sigma_1^+ \vdash \forall x\exists yT(e,x,y)$, hence $I\Sigma_1^+ \vdash \forall x\exists yT(e,x,y)$, and so by Parsons' theorem, there is a primitive recursive function symbol g such that $I\Sigma_1^+ \vdash \forall xT(e,x,gx)$. Invoking Π_2-conservativity of $I\Sigma_1^+$ over $iI\Sigma_1^+$, we obtain $iI\Sigma_1^+ \vdash \forall xT(e,x,gx)$. Now by the second conjunct of $e\,\mathbf{q}\forall x\exists y\varphi(x,y)$, we get $iI\Sigma_1^+ \vdash \forall xU(gx)\,\mathbf{q}\exists y\varphi(x,y)$, which is $\forall x(U(gx))_0\,\mathbf{q}\,\varphi(x,(U(gx))_1)$. By the usual property of $\mathbf{q}$-realizability, $iI\Sigma_1^+ \vdash \forall x\varphi(x,(U(gx))_1)$; so $f \equiv (U\circ g)_1$ does the job.

We list a number of further results. Whenever the usual proof of the corresponding fact for HA carries over literally to our case, we omit it.

Lemma 7. *For every negative formula φ there is a natural number n such that, provably in $iI\Sigma_1^+$,*

$$(n\,\mathbf{r}\,\varphi) \leftrightarrow \exists x(x\,\mathbf{r}\,\varphi) \leftrightarrow \varphi.$$

Lemma 8. *For every almost negative formula φ there is a p.r. term t, $FV(t) \subseteq FV(\varphi)$, such that*

$$iI\Sigma_1^+ \vdash \exists x(x\,\mathbf{r}\,\varphi) \leftrightarrow \exists y(t = y\,\&\,y\,\mathbf{r}\,\varphi).$$

Theorem 2. *For each instance Ψ of ECT_0 there is a p.r. term t such that*

$$iI\Sigma_1^+ \vdash \exists y(t = y\,\&\,y\,\mathbf{r}\,\Psi).$$

Corollary 2. *The following hold:*

1. $iI\Sigma_1^+ + ECT_0 \vdash \varphi \Rightarrow iI\Sigma_1^+ \vdash \exists x(x\,\mathbf{r}\,\varphi)$.
2. *For negative φ, $iI\Sigma_1^+ + ECT_0 \vdash \varphi \iff iI\Sigma_1^+ \vdash \varphi$.*
3. *$iI\Sigma_1^+ + ECT_0$ is consistent iff $iI\Sigma_1^+$ is.*
4. *For each formula φ, $iI\Sigma_1^+ + ECT_0 \vdash \varphi \leftrightarrow \exists x(x\,\mathbf{r}\,\varphi)$.*
5. *$iI\Sigma_1^+ + ECT_0$ has (DP) and (ED).*
6. *If $\forall x\exists y\varphi(x,y)$ is a sentence derivable in $iI\Sigma_1^+ + ECT_0$, then for some natural number e, $iI\Sigma_1^+ + ECT_0 \vdash \forall x\exists y(\{e\}(x) = y\,\&\,\varphi(x,y))$.*
7. *$iI\Sigma_1^+ + ECT_0$ is Π_2-conservative over $iI\Sigma_1^+$.*
8. *$iI\Sigma_3^+ + ECT_0$ is Π_2-conservative over $iI\Pi_2^+$.*
9. *If $iI\Sigma_1^+ + ECT_0$ proves a sentence $\forall x\exists y\varphi(x,y)$, then for some primitive recursive function symbol f, $iI\Sigma_1^+ + ECT_0 \vdash \forall x\varphi(x,fx)$ (and thus by the above, $iI\Sigma_1^+ \vdash \forall x\varphi(x,fx)$).*

Proof. '7.' Let $\varphi(x,y)$ be atomic and suppose $iI\Sigma_1^+ + ECT_0 \vdash \forall x\exists y\varphi(x,y)$. By 1., $iI\Sigma_1^+ \vdash \exists z(z\,\mathbf{r}\forall x\exists y\varphi(x,y))$. By (ED) for $iI\Sigma_1^+$, there is an $e \in \mathbb{N}$ such that $iI\Sigma_1^+ \vdash e\,\mathbf{r}\forall x\exists y\varphi(x,y)$. Thus $iI\Sigma_1^+ \vdash \forall x\exists z(T(e,x,z)\&\,Uz\,\mathbf{r}\exists y\varphi(x,y))$, so that $iI\Sigma_1^+ \vdash \forall x\exists z((Uz)_0\,\mathbf{r}\,\varphi(x,(Uz)_1))$, i.e. $iI\Sigma_1^+ \vdash \forall x\exists z\varphi(x,(Uz)_1)$ (φ being atomic), hence $iI\Sigma_1^+ \vdash \forall x\exists y\varphi(x,y)$.
'8'. Suppose $iI\Sigma_3^+ + ECT_0 \vdash \forall x\exists y\varphi(x,y)$ with φ atomic. By Theorem 2 and Lemma 6 we obtain $iI\Pi_2^+ \vdash \exists z(z\,\mathbf{r}\forall x\exists y\varphi(x,y))$. By (ED) for $iI\Pi_2^+$ there is an $e \in \mathbb{N}$ such that $iI\Pi_2^+ \vdash e\,\mathbf{r}\forall x\exists y\varphi(x,y)$, thus $iI\Pi_2^+ \vdash \forall x\exists z(T(e,x,z)\&$ $Uz\,\mathbf{r}\exists y\varphi(x,y))$, so that $iI\Pi_2^+ \vdash \forall x\exists z((Uz)_0\,\mathbf{r}\,\varphi(x,(Uz)_1))$, i.e. $iI\Pi_2^+ \vdash \forall x\exists z\varphi(x,(Uz)_1)$ (φ being atomic), hence $iI\Pi_2^+ \vdash \forall x\exists y\varphi(x,y)$.

'9'. If $iI\Sigma_1^+ + ECT_0 \vdash \forall x \exists y \varphi(x, y)$, where $\varphi(x, y)$ is an arbitrary formula having only x, y free, then by 6 for some natural number e, $iI\Sigma_1^+ + ECT_0 \vdash \forall x \exists z(T(e, x, z) \& \varphi(x, Uz))$. In particular $iI\Sigma_1^+ + ECT_0 \vdash \forall x \exists z T(e, x, z)$, so by 7 $iI\Sigma_1^+ \vdash \forall x \exists z T(e, x, z)$. By Parsons' Theorem, for some primitive recursive function symbol g, $iI\Sigma_1^+ \vdash \forall x T(e, x, gx)$. So clearly $iI\Sigma_1^+ + ECT_0 \vdash \forall x T(e, x, gx)$ and by properties of the T-predicate, $iI\Sigma_1^+ + ECT_0 \vdash \forall x(T(e, x, gx) \& \varphi(x, U gx))$. So $f = U \circ g$ does the job, q.e.d.

It is classically obvious that $PA = \bigcup_{n<\omega} I\Sigma_n$, i.e. induction over arbitrary prenex formulas already yields full *PA*. One might expect that a similar result can be obtained for intuitionistic arithmetic. However, this conjecture is dramatically false, as the following unpublished theorem of Visser's shows:

Theorem 3 (Visser). *Let iPNF be iPRA plus the induction schema for arbitrary prenex formulas. Then iPNF is Π_2-conservative over $iI\Pi_2^+$.*

Proof. For each formula ϕ we have $iI\Sigma_1^+ + ECT_0 \vdash \phi \leftrightarrow \exists x\, x \,\mathbf{r}\, \phi$ by 4. of Corollary 2.

If ϕ is prenex, then $z \,\mathbf{r}\, \phi$ is intuitionistically equivalent to a Π_2-formula, which we show by induction on the complexity of ϕ: If ϕ is atomic, there is nothing to show.

Suppose $\phi \equiv \forall x \psi(x)$. Then $z \,\mathbf{r} \forall x \psi(x) \equiv \forall x \exists y T(z, x, y) \& \forall xy(T(z, x, y) \to Uy \,\mathbf{r}\, \psi(x))$. The induction hypothesis yields that $Uy \,\mathbf{r}\, \psi(x))$ is equivalent to $\forall v \exists w \chi(v, w, x, y)$ for some quantifier-free formula χ. But then the formula $\forall xy(T(z, x, y) \to Uy \,\mathbf{r}\, \psi(x))$ is equivalent to $\forall xyv \exists w(T(z, x, y) \to \chi(v, w, x, y))$ which is Π_2. If $\phi \equiv \exists x \psi(x)$, then $z \,\mathbf{r}\, \phi$ is by definition $(z)_0 \,\mathbf{r}\, \psi((z)_1)$ which is Π_2 by induction hypothesis.

Hence every prenex formula is equivalent to a Σ_3-formula in $iI\Sigma_1^+ + ECT_0$, and thus $iPNF + ECT_0 \equiv iI\Sigma_3^+ + ECT_0$. But $iI\Sigma_3^+ + ECT_0$ is Π_2-conservative over $iI\Pi_2^+$ by Corollary 2, q.e.d.

Remark. Visser's original theorem stated the Π_2-conservativity of *iPNF* over $iI\Sigma_3^+$. The sharpening here is due to the author. In this form, Visser's theorem yields an exact characterization of the proof-theoretic strength of *iPNF* in terms of provably recursive functions:

Consider the Ackermann function, e.g in the following version (taken from [B93]): $\text{ack}(i, 0) = 2; \text{ack}(0, j+1) = \text{ack}(0, j)+2$ and $\text{ack}(i+1, j+1) = \text{ack}(i, \text{ack}(i+1, j))$. It is well known that this function is not primitive recursive and hence cannot be proved total in $iI\Sigma_1^+$. The usual proof of the totality of Ackermann's function in $I\Pi_2^+$ is constructive and can thus be carried out in $iI\Pi_2^+$. So *iPNF* is, while dramatically weaker than *IPNF*, still stronger than $iI\Sigma_1^+$, viz. of the same strength as $iI\Pi_2^+$.

Considering Visser's theorem on the weakness of the Σ_n-induction 'hierarchy' in the intuitionistic case, the question arises whether a useful hierarchy can be defined for intuitionistic arithmetic. One would certainly have to take some sort of 'implicational complexity' into account, in addition to the usual quantifier complexity, cf. [L81].

3 $iI\Sigma_1$ and $iI\Pi_1$

It is well-known that $I\Sigma_1$ and $I\Pi_1$ are equivalent classical theories, i.e. they have the same set of theorems (cf. [K91]). However, the standard proof of this fact involves an appeal to the principle of the excluded middle (or rather to Markov's Principle) and is thus not intuitionistically valid. In fact, the analogous statement for the intuitionistic case is completely false. Buss in his [Bu93] exhibits a Kripke model of $iI\Sigma_1$ which does not validate $iI\Pi_1$, so clearly $iI\Sigma_1 \nvdash iI\Pi_1$. We will show below that the natural conjecture $iI\Pi_1 \nvdash iI\Sigma_1$ is also true.

An analysis of the usual proof of $I\Sigma_1 \equiv I\Pi_1$ yields the following

Theorem 4. $iI\Pi_1 \equiv iI\neg\neg\Sigma_1$.

Proof. We first show that $iI\Pi_1 \vdash iI\neg\neg\Sigma_1$.
Let $\varphi(x,y)$ be a Δ_0-formula (possibly containing variables other than x,y free). Argue in $iI\Pi_1$ and assume $\forall x(\neg\neg\exists y\varphi(x,y) \to \neg\neg\exists y\varphi(x+1,y))$. In addition, suppose there is an a such that $\neg\exists y\varphi(a,y)$. Now we show that $\forall x[\forall z(x+z=a \to \forall y\neg\varphi(z,y)]$ by induction (note that the formula in square brackets is intuitionistically equivalent to a Π_1-formula):

If $x=0$ and z is such that $x+z=a$, clearly $z=a$. By assumption, $\neg\exists y\varphi(a,y)$ which is equivalent to $\forall y\neg\varphi(a,y)$.

For the induction step, assume $\forall z(x+z=a \to \forall y\neg\varphi(z,y)$ and let z be such that $(x+1)+z=a$. Then certainly $x+(z+1)=a$, so by the induction hypothesis, $\forall y\neg\varphi(z+1,y)$ which is $\neg\exists y\varphi(z+1,y)$. By our initial assumption that $\forall x(\neg\neg\exists y\varphi(x,y) \to \neg\neg\exists y\varphi(x+1,y))$, we obtain by contraposition (noting that intuitionistically $\neg\neg\neg A \leftrightarrow \neg A$) and instantiating x by z that $\neg\exists y\varphi(z,y)$, i.e. $\forall y\neg\varphi(z,y)$. Now letting $x=a$ we obtain $\neg\exists y\varphi(0,y)$. We have thus shown from $\forall x(\neg\neg\exists y\varphi(x,y) \to \neg\neg\exists y\varphi(x+1,y))$ that $\neg\exists\varphi(a,y) \to \neg\exists y\varphi(0,y)$, so by contraposition we get

$$\neg\neg\exists y\varphi(0,y)\,\&\,\forall x(\neg\neg\exists y\varphi(x,y) \to \neg\neg\exists y\varphi(x+1,y)) \to \neg\neg\exists y\varphi(a,y),$$

which practically is the induction axiom for $\neg\neg\exists y\varphi(x,y)$.

Now we show that $iI\neg\neg\Sigma_1 \vdash iI\Pi_1$.

First note that $I\Sigma_1 \vdash A$ entails $iI\neg\neg\Sigma_1 \vdash A^-$ (where A^- is the negative translation of A) since the induction axioms of $iI\neg\neg\Sigma_1$ are precisely the negative translations of the Σ_1-induction axioms.

So since $I\Sigma_1 \vdash B\Sigma_1$ (the collection axioms for Σ_1-formulas, cf. [K91]), we have, for any Δ_0-formula φ:

$iI\neg\neg\Sigma_1 \vdash \forall x \leq a\neg\neg\exists y\varphi(x,y) \leftrightarrow \neg\neg\exists z\forall x \leq a\exists y \leq z\varphi(x,y)$. Hence in $iI\neg\neg\Sigma_1$, the $\neg\neg\Sigma_1$-formulas are closed under bounded universal quantification.

Now let $\varphi(x,y)$ be a Δ_0-formula. We will argue in $iI\neg\neg\Sigma_1$.

Suppose $\forall x(\forall y\varphi(x,y) \to \forall y\varphi(x+1,y))$ and assume $\neg\forall y\varphi(a,y)$. We want to show that

$$(*) \quad \forall x\forall z \leq a(x+z=a \to \neg\neg\exists y\neg\varphi(z,y)).$$

In intuitionistic predicate logic, the formula in parentheses is equivalent to $\neg\neg(x+z=a \to \exists y\neg\varphi(z,y))$, which is (in $iI\Delta_0$) equivalent to the formula $\neg\neg\exists y(x+$

$z = a \to \neg\varphi(z, y))$ since $x + z = a$ is decidable. Hence $(*)$ is equivalent to $\forall x[\forall z \leq a \neg\neg\exists y(x + z = a \to \neg\varphi(z, y))]$, and by the remark above, the formula in square brackets is $\neg\neg\Sigma_1(iI\neg\neg\Sigma_1)$. Hence we may apply $\neg\neg\Sigma_1$-induction to prove $(*)$. This works as before.

Corollary 3. *The following obtain:*

1. $iI\Sigma_1 + MP_{PR} \equiv iI\Pi_1 + MP_{PR}$.
2. *The negative translation does not work for $I\Sigma_1$ and $iI\Sigma_1$, i.e. there are formulas A such that $I\Sigma_1 \vdash A$ and $iI\Sigma_1 \nvdash A^-$.*
3. *$iI\Sigma_1$ and $iI\Pi_1$ prove the same Π_1-sentences.*
4. $iI\Sigma_1 \vdash Th_{\Pi_2}(iI\Pi_1)$.

Proof. 1 is clear. If 2 were false, then $iI\Sigma_1$ would prove the negative translations of the Σ_1-induction axioms and thus $iI\Sigma_1 \vdash iI\neg\neg\Sigma_1$, contradicting Buss' result. If $iI\Sigma_1 \vdash \forall\bar{x}\varphi(\bar{x})$, where $\varphi \in \Delta_0$, then $I\Sigma_1 \vdash \forall\bar{x}\varphi(\bar{x})$ and hence $iI\neg\neg\Sigma_1$ proves $(\forall\bar{x}\varphi(\bar{x}))^-$, which is just $\forall\bar{x}\varphi(\bar{x})$. If $iI\Pi_1 \vdash \forall x\exists y\varphi(x, y)$, so does $I\Pi_1$ and thus $I\Sigma_1$ which is Π_2-conservative over $iI\Sigma_1$.

We will now establish, by model-theoretic means, that $iI\Pi_1$ cannot prove $iI\Sigma_1$; in fact we will show that the provably recursive functions of $iI\Pi_1$ are all majorized by polynomials over $\mathbb{N}$. First, some preparatory lemmas:

Lemma 9. *Let M be a classical model of $I\Delta_0$. Then the following conditions are equivalent:*

1. *There is a Δ_0-elementary extension N of M such that $N \models I\Sigma_1$.*
2. $M \models Th_{\Pi_1}(I\Sigma_1)$.

Proof. $1 \Rightarrow 2$ follows from the fact that Π_1-sentences are preserved downwards in Δ_0-elementary extensions. So suppose that $M \models Th_{\Pi_1}(I\Sigma_1)$. Let T be the theory axiomatized by $I\Sigma_1$ plus the Δ_0-diagram of M. We are done if we can show T has a model. Otherwise, by compactness there are finitely many Δ_0 sentences $\gamma_1(\bar{c}), ..., \gamma_n(\bar{c})$ true in M (all parameters $\bar{c}$ from M indicated) such that $I\Sigma_1 \vdash \neg\bigwedge\gamma_i(\bar{c})$, so by the lemma on new constants, $I\Sigma_1 \vdash \forall\bar{x}\neg\bigwedge\gamma_i(\bar{x})$. Hence this last sentence is in $Th_{\Pi_1}(I\Sigma_1)$, which is impossible since $M \models \bigwedge\gamma_i(\bar{c})$.

Fortunately it is not difficult to construct Kripke models of $iI\Pi_1$, as the next lemma shows. In general it is a rather hard task to construct such models for more complex theories, or even to state what classical structures such models are composed of, see e.g. [S73], [Bu93], [M93], [W?].

Lemma 10. *Let $\mathbf{K} = (\{0, 1\}, \leq, (A_0, A_1))$ be a Kripke structure such that $A_0 \models I\Delta_0$ and $A_1 \models I\Sigma_1$, and A_1 is a Δ_0-elementary extension of A_0. Then $\mathbf{K} \models iI\Pi_1$.*

Proof. It can be verified directly that $0 \Vdash iI\Pi_1$. Alternatively, observe that the terminal node 1 forces $iI\Sigma_1$, so 0 forces $\neg\neg iI\Sigma_1$. Obviously, the schema (DNS) (double negation shift) $\neg\neg\forall x\psi \to \forall x\neg\neg\psi$ is forced at 0 due to the simple shape of the Kripke model, so that, at 0, $\neg\neg iI\Sigma_1$ is $iI\neg\neg\Sigma_1$, which is $iI\Pi_1$, q.e.d.

Theorem 5. *If $iI\Pi_1 \vdash \forall \overline{x} \exists \overline{y} \phi(\overline{x}, \overline{y})$ with $\phi \in \Delta_0$, then there is a polynomial $p(\overline{x})$ with coefficients in $\mathbb{N}$ such that $\mathbb{N} \models \forall \overline{x} \exists \overline{y} \leq p(\overline{x}) \phi(\overline{x}, \overline{y})$.*

Proof. If $iI\Pi_1 \vdash \forall \overline{x} \exists \overline{y} \phi(\overline{x}, \overline{y})$, every Kripke model of the sort described in the previous lemma validates this Π_2-sentence, and as is easy to see, the sentence will thus be true in the classical model of $I\Delta_0$ attached to the bottom node. So by Lemma 8, $\forall \overline{x} \exists \overline{y} \phi(\overline{x}, \overline{y})$ will hold in every model of $I\Delta_0 + Th_{\Pi_1}(I\Sigma_1)$ and hence be provable in this theory. A well-known proof-theoretical argument shows that every provably recursive function of $I\Delta_0$ is majorized by some polynomial; the same proof works for $I\Delta_0$ augmented by any set of true Π_1-sentences, in particular for $I\Delta_0 + Th_{\Pi_1}(I\Sigma_1)$. Our claim follows.

Let us establish some corollaries.

Corollary 4. *$iI\Pi_1 \nvdash iI\Sigma_1$.*

Proof. The exponentiation function is provably total in $iI\Sigma_1$.

Corollary 5. *$iI\Pi_1$ is not closed under MR_{PR}.*

Proof. If $I\Sigma_1 \vdash \forall x \exists y \varphi(x, y)$ with φ in Δ_0 having only x, y free, then $iI\neg\neg\Sigma_1 \vdash \neg\neg\exists y \varphi(x, y)$ since the negative translation obviously works for the theories $I\Sigma_1$ and $iI\neg\neg\Sigma_1$, the translations of Σ_1-induction axioms being the $\neg\neg\Sigma_1$- induction axioms due to $\neg\forall x \neg\psi \leftrightarrow \neg\neg\exists x \psi$. If $iI\neg\neg\Sigma_1 \equiv iI\Pi_1$ were closed under MR_{PR}, $I\Sigma_1$ would be Π_2-conservative over $iI\Pi_1$ which is not the case.

Corollary 6. *The following theories are all equivalent, and none of them implies $iI\Sigma_1$ or is implied by $iI\Sigma_1$: $iI\Pi_1$, $iI\neg\neg\Sigma_1$, $iI\neg\Sigma_1$, $iI\neg\Pi_1$, $iI\neg\neg\Pi_1$.*

Proof. Using $iI\Sigma_1 \nvdash iI\Pi_1$, $iI\Pi_1 \nvdash iI\Sigma_1$, $iI\Pi_1 \equiv iI\neg\neg\Sigma_1$, the results follow easily by employing decidability of Δ_0-formulas in $iI\Delta_0$ and the identities $\neg\exists x A \leftrightarrow \forall x \neg A$, $\neg\neg\neg A \leftrightarrow \neg A$.

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References

[B93] Boolos, G.: The Logic of Provability. Cambridge: Cambridge University Press 1993

[Bu93] Buss, S.: Intuitionistic validity in T-normal Kripke structures. Ann. Pure Appl. Logic **59**, 159–173 (1993)

[D94] Damnjanovic, Z.: Strictly primitive recursive realizability, I. JSL **59**, No. 4, 1210–1227 (1994)

[Dr87] Dragalin, A.: Mathematical Intuitionism – Introduction to Proof Theory. Providence, RI: Am. Math. Soc. 1987

[K91] Kaye, R.: Models of Peano Arithmetic. Oxford: Oxford University Press 1991

[L81] Leivant, D.: Implicational complexity in intuitionistic arithmetic. JSL **46**, 240–248 (1981)

[M93] Markovic, Z.: On the structure of Kripke models of Heyting arithmetic. MLQ **39**, 531–538 (1993)

[P92] Pohlers, W.: A short course in ordinal analysis. In: Aczel, P. et al. (ed.) Proof Theory, pp. 27–79 Cambridge: Cambridge University Press 1992

[S73] Smorynski, C.: Applications of Kripke models. In: [T73], pp. 324–391

[T73] Troelstra, A.S. (ed.): Metamathematical Investigations of Intuitionistic Arithmetic and Analysis. Berlin Heidelberg New York: Springer 1973

[T92] Troelstra, A.S.: Realizability. ILLC Prepublication Series, ML-92-09, Amsterdam: (1992)

[TVD88] Troelstra, A.S., van Dalen, D.: Constructivism in Mathematics, Vol. I. Amsterdam: North-Holland 1988

[VMKV86] van Dalen, D., Mulder, H., Krabbe, E.C.W., Visser, A.: Finite Kripke Models of HA are locally PA. Notre Dame J. Formal Logic **27**, 528–532 (1986)

[W?] Wehmeier, K.F.: Classical and Intuitionistic Models of Arithmetic. Notre Dame J. Formal Logic (to appear)