WITTGENSTEINIAN TABLEAUX, IDENTITY, AND CO-DENOTATION

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Abstract. Wittgensteinian predicate logic (W-logic) is characterized by the fact that the objects mentioned within the scope of a quantifier are excluded from the range of the associated bound variable. I present a sound and complete tableaux calculus for this logic and discuss issues of translatability between Wittgensteinian and standard predicate logic in languages with and without individual constants. A metalinguistic co-denotation predicate, akin to Frege's triple bar of the Begriffsschrift, is introduced and used to bestow the full expressive power of first-order logic with identity on W-logic in the presence of constants.

1. Introduction

Wittgensteinian predicate logic (W-logic), first systematically investigated by Hintikka [8], is characterized by the fact that all objects mentioned within the scope of a quantifier are excluded from the range of the associated bound variable. Thus, for instance, the sentence $\forall x Rxa$ asserts, in W-logic, that all individuals *other than* a bear the relation R to a; similarly, the sentence $\forall x \exists y Rxy$ says that for every individual x, there exists an individual y other than x such that x bears R to y.

This logic is inspired by certain remarks of Wittgenstein's (5.53ff) in the Tractatus [18], but can also be motivated independently. Consider the following arguments.[1]

(1) I agree with myself. Therefore, somebody agrees with me.
(2) Everybody applauded him. Therefore, he applauded himself.

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Neither of these arguments seem to be intuitively valid; yet standard semantics for the quantifiers would seem to make them so. On a Wittgensteinian reading of the quantifiers, the invalidity of these arguments becomes readily apparent: In 1, the referent of 'me' is exempted from the range of the existential quantifier in the conclusion, and in 2, the referent of 'him' is excluded from the range of the universal quantifier in the premise.

As Hintikka has shown, Wittgenstein's ideas can indeed be worked out in such a way as to provide a predicate logic, without a symbol for identity, that has the same expressive power as ordinary first-order logic with identity ($FOL^=$)–as long as no individual constants and no function symbols are present in the language.

Hintikka asserts without proof that a calculus he describes in [8] is sound and complete for W-logic. This calculus is, however, of a rather nonstandard sort. In [16] I formulated a Gentzen-style sequent calculus and proved it sound and complete with respect to the semantics of W-logic. In the present paper, I describe a sound and complete *tableaux* calculus for W-logic; it turns out that this calculus is very similar, except for the obvious constraints on the quantifier rules, to a calculus first suggested by John Burgess for first-order logic FOL (first discussed in print by Boolos [3]; see also the latest edition of the major introductory textbook [2]; and for its application to languages with identity and function symbols, see Bergmann [1]).

The issue of individual constants in W-logic has, as far as I am aware, never been discussed in the literature. It is shown here that constants may be added to the underlying first-order language without jeopardizing the mutual translatability of W-logic and $FOL^=$ if, and only if, no two constants are ever permitted to have the same value. Moreover, by adding a relation of co-denotation to the language of W-logic, the mutual translatability of $FOL^=$ and W-logic can be restored even in the presence of possibly co-denoting constants. As a byproduct of these results, we also show that identity is equivalent, over FOL without function symbols, to the collection of all finite numerical quantifiers (i.e., quantifiers of the sort 'there are at least n things x such that ...').

Wittgenstein's motivation in denying identity the status of a relation appears to be his belief that truth-functional tautologies are the sole source of logical necessity. Clearly, the identity relation violates this principle in that identities would be atomic facts whose necessity is not rooted in tautologies (cf. Williams [17, pp. 24-25]).

But there are arguments against the relational nature of identity that are independent of specifically Tractarian doctrines. For example, it seems natural to define the arity of a relation as the (maximal) number of objects it can relate. On this proposal identity turns out to be a (unary) property, as it never holds of two objects (which would seem to be Wittgenstein's point in 5.5303).

For another argument (noted by Pardey [11, p. 38]), consider the following analogy between non-existence and identity: A singular statement of the form 'a does not exist' is about either one or no object, depending on its truth. Similarly, a statement of the form '$a{=}b$' is about either two objects or one, depending on its truth. Now if the former observation regarding (non-)existence suggests that existence is not a property of individuals, then the latter might likewise suggest that identity is not a relation between individuals.

These brief argument sketches are obviously subject to objections; and it is not my aim here to argue that identity is not a relation (although I do believe that this claim needs to be taken seriously). I mention them only by way of philosophical motivation for the logical study that follows.

From a philosophically neutral position, the mutual translatability of W-logic and FOL$^=$ can be taken to show how the content of the identity symbol in first-order logic can be carved up in various ways, and be distributed between quantifiers and variables (as well as, where appropriate, a co-denotation predicate) so as to render '=' superfluous.

From the less neutral vantage point of someone wishing to argue that identity is not a relation, one might use the interpretability of FOL$^=$ in W-logic to defuse the objection that banning the identity symbol leads to a crippling of logic's expressive power. To be sure, the mere elimination of identity from FOL$^=$ leads to the weaker logic FOL, in which we can no longer express things like 'there are exactly two objects' or 'the F is a G'. The opponent of identity can now argue, however, that a slight adjustment of the semantics for the quantifiers and variables (which can be independently motivated) restores full expressive power.[2]

2. Semantics for Wittgensteinian Predicate Logic

The language $\mathcal{L}$ of first-order logic, as understood here, has the following primitive symbols.

(1) the usual propositional connectives
(2) the quantifier symbols $\forall$ and $\exists$
(3) countably many variables $x, y, x_1, x_2, \ldots$
(4) countably many parameters $a, b, a_1, a_2, \ldots$
(5) for each positive integer n, countably many n-ary predicate symbols $P^n, Q^n, R^n, P^n_1, P^n_2, \ldots$

An *individual symbol* is either a variable or a parameter. Atomic formulas are constructed from predicates and individual symbols as usual, and compound formulas are built up inductively by means of the propositional connectives and the quantifiers as usual. A *pure formula* is one in which no parameters occur (in section 4, we will consider languages with individual constants, where pure formulas will still contain no parameters but are allowed to contain constants). A *sentence* or *closed formula* is a formula in which no variable has a free occurrence. A *pure sentence*, accordingly, is a formula with neither occurrences of parameters nor free occurrences of variables. Where U is any non-empty set, the *U-formulas* are defined just like the ordinary formulas, except that elements of U are used instead of parameters. The *U-sentences* (or *closed U-formulas*) are U-formulas without free occurrences of variables.[3]

A structure $\mathcal{U}$ is a non-empty domain U together with an n-ary relation $R_{\mathcal{U}}$ over U for each n-ary predicate symbol R of $\mathcal{L}$.

We now define a notion of *W-truth* in a structure $\mathcal{U}$ for pure sentences A; this will be done by defining such a notion for the U-sentences, of which the pure sentences form a subset.

Definition 1. *We inductively define a binary relation $\mathcal{U} \Vdash A$ between structures $\mathcal{U}$ and U-sentences A of $\mathcal{L}$ as follows.*

(1) *For $u_1, \ldots, u_n \in U$, $\mathcal{U} \Vdash Ru_1 \ldots u_n$ if $\langle u_1, \ldots, u_n \rangle \in R_{\mathcal{U}}$.*
(2) *$\mathcal{U} \Vdash \neg B$ if $\mathcal{U} \nVdash B$.*
(3) *$\mathcal{U} \Vdash B \to C$ if $\mathcal{U} \nVdash B$ or $\mathcal{U} \Vdash C$. Similarly for the other Boolean connectives.*
(4) *$\mathcal{U} \Vdash \forall xF$ if for every $u \in U$ other than those occurring in $\forall xF$, $\mathcal{U} \Vdash F^x_u$ (where F^x_u is the result of substituting u for every free occurrence of x in F).*
(5) *$\mathcal{U} \Vdash \exists xF$ if for some $u \in U$ other than those occurring in $\exists xF$, $\mathcal{U} \Vdash F^x_u$.*

In preparation for the soundness and completeness proofs, we define notions of W-satisfiability and W-validity for sentences containing parameters (and sets of such sentences) as follows.

Definition 2. *Let F be a sentence, and let V be a set of parameters including those occurring in F. Let σ be a function from V to U. We say that σ W-satisfies F in $\mathcal{U}$ if F^σ, i.e. the result of simultaneously replacing, in F, every parameter a by $\sigma(a)$, is W-true in $\mathcal{U}$.*

F is W-satisfiable in $\mathcal{U}$ if there exists a 1-1 *function σ, defined at least on the parameters in F and with values in U, such that σ W-satisfies F in $\mathcal{U}$.*

F is W-valid in $\mathcal{U}$, $\mathcal{U} \Vdash F$, if every 1-1 *function σ, defined at least on the parameters in F and with values in U, W-satisfies F in $\mathcal{U}$.*

F is W-satisfiable if it is W-satisfiable in $\mathcal{U}$ for some $\mathcal{U}$.

F is W-valid, $\Vdash F$, if it is W-valid in every $\mathcal{U}$.

If S is a (finite or infinite) set of sentences, and V a set of parameters including those occurring in elements of S, then $\sigma : V \to U$ W-satisfies S in $\mathcal{U}$ if σ W-satisfies every member of S in $\mathcal{U}$. S is W-satisfiable in $\mathcal{U}$ if, for some set V of parameters including those occurring in elements of S, there is some 1-1 *function $\sigma : V \to U$ such that σ W-satisfies S in $\mathcal{U}$, etc.*

As was first pointed out by Hintikka, the logic thus defined (which, following [16], we refer to as Wittgensteinian or W-logic) is intertranslatable with FOL$^=$.[4]

Inductively define a function ψ from the $\mathcal{L}$-sentences to the $\mathcal{L}^=$-sentences as follows. For n-ary predicate symbols R, $\psi(Ra_1 \dots a_n)$ is just $Ra_1 \dots a_n$. For Boolean connectives, ψ acts like a homomorphism (e.g. $\psi(A \wedge B)$ is $\psi(A) \wedge \psi(B)$). $\psi(\forall x F)$ is

$$\forall x \left(\bigwedge_{i=1}^{n} x \neq a_i \to \psi(F) \right),$$

where $a_1, \dots, a_n$ are precisely the parameters occurring in $\forall x F$, and $\psi(F)$ is the result $\psi(F^x_a)^a_x$ of replacing a with x in the sentence $\psi(F^x_a)$, a being a fresh parameter. And finally, $\psi(\exists x F)$ is

$$\exists x \left(\bigwedge_{i=1}^{n} x \neq a_i \wedge \psi(F) \right),$$

where again $a_1, \dots a_n$ are precisely the parameters occurring in $\exists x F$, and $\psi(F)$ is as above.[5]

Conversely, define a function φ from the $\mathcal{L}^=$-sentences to the $\mathcal{L}$-sentences inductively as follows. For n-ary predicate symbols R other than the equality symbol, $\varphi(Ra_1 \dots a_n)$ is just $Ra_1 \dots a_n$. $\varphi(a = a)$ is $Pa \to Pa$, for some designated unary predicate symbol P. For distinct parameters a and b, we let $\varphi(a = b)$ be $\neg(Rab \to Rab)$, for some designated binary predicate symbol R. For Boolean connectives, φ acts like a homomorphism. $\varphi(\forall x F)$ is

$$\forall x \varphi(F) \wedge \bigwedge_{i=1}^{n} \varphi(F^x_{a_i}),$$

where $a_1, \dots a_n$ are precisely the parameters occurring in $\forall x F$, and $\varphi(F)$ is the result $\varphi(F^x_a)^a_x$ of replacing a with x in the sentence $\varphi(F^x_a)$,

a being a fresh parameter. Finally, $\varphi(\exists x F)$ is

$$\exists x \varphi(F) \vee \bigvee_{i=1}^{n} \varphi(F^{x}_{a_i}),$$

where $a_1, \ldots a_n$ are precisely the parameters occurring in $\exists x F$ and $\varphi(F)$ is as above.[6]

One can then show that, for pure $\mathcal{L}$-sentences A and pure $\mathcal{L}^{=}$-sentences B, $\mathcal{U} \Vdash A$ iff $\mathcal{U} \models \psi(A)$ and $\mathcal{U} \models B$ iff $\mathcal{U} \Vdash \varphi(B)$, where $\models$ is the usual relation of truth-in-a-model for FOL$^{=}$. Moreover, for pure $\mathcal{L}$-sentences A and pure $\mathcal{L}^{=}$-sentences B, $\Vdash A \leftrightarrow \varphi(\psi(A))$ and $\models B \leftrightarrow \psi(\varphi(B))$. W-logic and FOL$^{=}$ therefore have the same expressive power.

The intertranslatability of FOL$^{=}$ with W-logic implies the equivalence of FOL$^{=}$ and FOL augmented by the finite numerical quantifiers: Add to the language of FOL all quantifiers of the form $\exists^{\geq n}$ with $n \geq 1$. Syntactically, these function just like the ordinary quantifiers. Interpret a U-sentence $\exists^{\geq n} x F$ in a structure $\mathcal{U}$ as 'there are at least n elements u of U such that $\mathcal{U} \models F^{x}_{u}$'. Call the resulting logic FOL$_{num}$. As is well-known, FOL$_{num}$ can be interpreted in FOL$^{=}$. But the converse is also true:

Corollary 1. *For every pure sentence F of* FOL$^{=}$*, there is a sentence G of* FOL$_{num}$ *such that, for every structure $\mathcal{U}$, $\mathcal{U} \models F$ iff $\mathcal{U} \models G$.*

Towards a proof, note that it suffices to interpret W-logic, rather than FOL$^{=}$, in FOL$_{num}$, as we have already established the interpretability of FOL$^{=}$ in W-logic. We can then prove the corollary by first showing the following slightly more general result: For every sentence F (possibly containing parameters) of W-logic, there exists a sentence $\eta(F)$ of FOL$_{num}$ such that, for every structure $\mathcal{U}$ and every U-assignment σ that is 1-1 on the parameters of F, $\mathcal{U} \Vdash F^{\sigma}$ iff $\mathcal{U} \models \eta(F)^{\sigma}$. The translation function η can be defined inductively as follows. On atomic sentences, η is the identity function. With respect to Boolean connectives, η acts as a homomorphism. For the case of the quantifiers, we restrict ourselves here to a representative example (the general procedure should be readily apparent from it): Consider an $\mathcal{L}$-sentence $\exists x F$ that contains precisely the two parameters a and b. Its translation $\eta(\exists x F)$ into FOL$^{=}{}_{num}$ will be

$$\big(\eta(F^{x}_{a}) \wedge \eta(F^{x}_{b}) \wedge \exists^{\geq 3} x\, \eta(F)\big) \quad \vee \quad \big(\eta(F^{x}_{a}) \wedge \neg\eta(F^{x}_{b}) \wedge \exists^{\geq 2} x\, \eta(F)\big) \vee$$
$$\big(\neg\eta(F^{x}_{a}) \wedge \eta(F^{x}_{b}) \wedge \exists^{\geq 2} x\, \eta(F)\big) \quad \vee \quad (\neg\eta(F^{x}_{a}) \wedge \neg\eta(F^{x}_{b}) \wedge \exists x\, \eta(F))$$

Thus, $\eta(\exists x F)$ simply spells out all the possible combinations of a and b satisfying or not satisfying F, and uses numerical quantifiers to ensure that, in each disjunct, the existence of an object other than a and b satisfying F is asserted. Since we are restricted to 1-1 assignments, a and b must have distinct values are hence invariably counted as two. Restricting this result to pure sentences, we obtain the desired result, as all variable assignments are 1-1 on the empty set of parameters.

3. A Tableaux Calculus for W-Logic

Just as is the case with FOL, there is an elegant tableaux calculus for W-logic, which we describe in this section; we shall call it *the W-procedure*. It should be noted that the mere existence of a sound and complete calculus for W-logic comes as no surprise at all, given the intertranslatability with FOL$^=$. But the precise nature of the tableaux system is rather interesting and testifies to the intuitiveness of W-logic.

The propositional rules of the W-procedure, as well as the rules for negated quantified sentences, are standard, see e.g. Jeffrey [9], Smullyan [15], or Bergmann et al. [2]. The quantifier rules for W-logic are as follows.[7]

(W-EI) Suppose that the sentence $\exists x F$ occurs on an open branch B of a tree. Let $b_1, \ldots, b_k$ be the parameters occurring in sentences on B *other than those occurring in* $\exists x F$, and let a be some parameter not occurring anywhere on B. Then the open branch B may be extended into $k+1$ new branches by affixing $F^x_{b_1}, \ldots, F^x_{b_k}, F^x_a$, respectively, to the bottom of B.

(W-UI) Suppose that the sentence $\forall x F$ is on an open branch B of a tree. If a parameter a appears on the branch, *but not in* $\forall x F$, then the branch B may be extended by writing F^x_a at the bottom of B. If no parameters occur on B, the branch B may be extended by writing F^x_a at the bottom of B, where a is an arbitrarily chosen parameter. (If there are parameters occurring on B, but they all occur in $\forall x F$ as well, the rule is not applicable.)

As usual, we call a branch *closed* if it contains both an atomic sentence and its negation; a branch that is not closed is *open*. The soundness theorem for our tableaux calculus asserts that, if there exists a closed tree with a sentence A at its origin, then A is W-unsatisfiable. This follows easily from the following lemma.

Lemma 1 (Soundness Lemma)**.** *Let B be an open branch through a tree T constructed in finitely many steps according to the W-procedure, and let V be the set of all parameters occurring in sentences on B. Suppose that (the set of all sentences occurring on) B is W-satisfied in $\mathcal{U}$ by some* 1-1 *function $\sigma : V \to U$. Suppose further that T is extended* *to T' by applying one of the rules of the W-procedure. Then at least one of the branches through the new tree that contain B is W-satisfied in $\mathcal{U}$ by some* 1-1 *extension of σ.*

The interesting cases are those where T' is obtained from T by one of the quantifier rules. Consider (W-UI) first. If it was applied to a sentence $\forall x F$ on B, then we know that σ W-satisfies this sentence in $\mathcal{U}$, i.e. $\mathcal{U} \Vdash \forall x F^\sigma$.

Hence, for every $u \in U$ that is not an image under σ of a parameter occurring in $\forall x F$, $\mathcal{U} \Vdash (F^x_u)^\sigma$.

Now if the parameter a is in V, but doesn't occur in $\forall x F$, then it follows immediately that $\mathcal{U} \Vdash (F^x_{\sigma(a)})^\sigma$, i.e. $\mathcal{U} \Vdash (F^x_a)^\sigma$, which was to be shown.

But if no parameters occur on B, then V is empty, σ is the empty function, and no $u \in U$ is an image under σ of a parameter occurring in $\forall x F$. Since U is non-empty, we can choose an arbitrary element $u \in U$, and the 1-1 extension $\sigma' := \{\langle a, u\rangle\}$ of σ must W-satisfy F^x_a in $\mathcal{U}$, because $(F^x_u)^\sigma$ is $(F^x_a)^{\sigma'}$.[8]

Next consider (W-EI). If it was applied to a sentence $\exists x F$ on B, then we know that σ W-satisfies that sentence in $\mathcal{U}$, i.e. $\mathcal{U} \Vdash \exists x F^\sigma$. Hence, for some $u \in U$ that is not an image under σ of a parameter occurring in $\exists x F$, $\mathcal{U} \Vdash (F^x_u)^\sigma$.

Now if u is $\sigma(b_i)$ for some parameter $b_i \in V$ that doesn't occur in $\exists x F$, then σ itself W-satisfies every sentence on B as well as $F^x_{b_i}$.

Otherwise, u is an element of $U \setminus \sigma[V]$, and hence the extension of σ that maps the fresh parameter a to u not only W-satisfies F^x_a in $\mathcal{U}$, but is also 1-1 on $V \cup \{a\}$, which concludes the proof of lemma 1.

From lemma 1, the soundness theorem follows along standard lines: If one begins a tree with a W-satisfiable sentence, then at every stage one will have at least one W-satisfiable branch, and so the tree can never close.[9]

In turning now to the completeness theorem, let us call an open branch through a tree constructed according to the W-procedure a complete open branch if, informally, all the information on that branch has been used. In other words, for an open branch B to be complete, we require that:

(1) for every conjunction $F \wedge G$ occurring on B, both F and G also occur on B, and similarly for the other propositional connectives
(2) for every negated conjunction $\neg(F \wedge G)$, at least one of $\neg F$ and $\neg G$ also occur on B, and similarly for the other propositional connectives

(3) for every negated universal sentence $\neg\forall xF$ occurring on B, the sentence $\exists x\neg F$ also occurs on B, and similarly for negated existential sentences
(4) for every universal sentence $\forall xF$ on B: there are parameters a occurring in sentences on B, but not in $\forall xF$, and for each such parameter a, the sentence F^x_a also occurs on B
(5) for every existentially quantified sentence $\exists xF$ occurring on B, some sentence of the form F^x_a, where a does not occur in $\exists xF$, also occurs on B.

We have the following.

Lemma 2 (Completeness Lemma)**.** *Let B be a (finite or infinite) complete open branch in a tree T constructed according to the W-procedure. Let V be the set of parameters occurring in sentences on B. Then there exists a structure $\mathcal{U}$ and a* 1-1 *function $\sigma : V \to U$ such that σ W-satisfies every sentence on B in $\mathcal{U}$.*

For the proof, we let the domain of $\mathcal{U}$ be V, and define σ to be the identity function on V. For an n-ary predicate symbol R, we let $\langle a_1, \ldots, a_n\rangle \in R_{\mathcal{U}}$ if and only if the sentence $Ra_1 \ldots a_n$ occurs on B.

We claim that, for every sentence A, if A (respectively, $\neg A$) occurs on B, then $\mathcal{U} \Vdash A$ (respectively, $\mathcal{U} \nVdash A$). This is established by induction on A; the interesting cases are those where A is a quantified sentence.

Suppose first that A is $\forall xF$. Since B is complete, we know that for any parameter a occurring on B, but not in $\forall xF$, F^x_a occurs on B. Hence, by the inductive hypothesis, $\mathcal{U} \Vdash F^x_a$ for every a in V other than those occurring in $\forall xF$, i.e. $\mathcal{U} \Vdash \forall xF$.

Now consider the case where A is $\exists xF$. Since B is complete, some sentence of the form F^x_a, where a does not occur in $\exists xF$, also occurs on B. By the inductive hypothesis, $\mathcal{U} \Vdash F^x_a$; hence $\mathcal{U} \Vdash \exists xF$. This completes the proof of the lemma.

To obtain the completeness theorem from lemma 2, it remains to provide a systematic procedure for generating W-tableaux such that every tree constructed according to it will either close or contain a complete open branch. In this regard, there is nothing special about W-logic; any of the usual procedures will do (the crucial point is to always come back to universally quantified sentences and reapply (W-UI) in case additional parameters have been introduced at a later stage).

Readers familiar with Boolos [3] will note that the W-procedure is, except for the additional constraints on the instantiating parameters, identical to the tableaux calculus discussed there. In particular, our rule (W-EI) arises from the Burgess-Boolos modification of the standard existential instantiation rule by insisting that the instantiating parameters not occur in the existential sentence itself.[10] The rationale for replacing standard existential instantiation with the Burgess-Boolos rule was to ensure that the tableaux calculus have the finite tree property, that is, the property that a tableau begun with a sentence satisfiable in a finite domain will result in a finite complete open branch after a finite number of rule applications (and thus produce a finite model for the sentence). It should thus be no surprise that the W-procedure also has the finite tree property: If the construction of a tree is begun, at the origin, with a sentence W-satisfiable in a finite domain, then by systematic application of the W-rules, the construction will, after finitely many steps, produce a finite completed open branch. This is obvious from the proof of lemma 1, which shows that W-satisfiability of a tree in a given structure $\mathcal{U}$ (which in this case we may take to be finite) is preserved under extensions of trees by W-rules. But if $\mathcal{U}$ has n elements, then no branch W-satisfiable in $\mathcal{U}$ can contain more than n parameters (as the assignment function from parameters into the domain must be 1-1). Hence, after a finite number of steps, we are left with a complete open branch. By lemma 2, this branch determines a finite model of the tree's origin. Thus, whenever a sentence A has finite models, the W-procedure will produce one.[11]

4. Individual Constants

So far, we have been considering languages without any individual constants.[12] We have seen that, when formulated in such languages, W-logic and FOL$^=$ are fully equivalent.

In this section, we will establish the following facts. First, W-logic and FOL$^=$ remain fully intertranslatable even with respect to languages containing constants, provided that structures never interpret distinct constants as the same element of their domain. Second, if no such restriction is placed on the interpretation of constants, then FOL$^=$ is no longer completely interpretable in W-logic. In the next section, we will show how the addition of a co-denotation predicate to W-logic remedies this defect and restores W-logic's expressive power to that of FOL$^=$ even in the presence of possibly co-denoting constants.

The syntax of W-logic with constants is simply that of FOL with those same constants. Structures $\mathcal{U}$ now come equipped with interpretations $c^{\mathcal{U}} \in U$ for each constant c in the language. We extend the definition of W-truth as follows: $\mathcal{U} \Vdash Rt_1 \dots t_n$ iff $\langle \mathcal{U}(t_1), \dots, \mathcal{U}(t_n) \rangle \in R^{\mathcal{U}}$, where each t_i is either an element u of $\mathcal{U}$ (in which case $\mathcal{U}(u)$ is defined to be just u) or an individual constant c (in which case $\mathcal{U}(c)$ is defined to be $c^{\mathcal{U}}$). $\mathcal{U} \Vdash \forall x F$ iff for every u in U other than those mentioned in $\forall x F$ (where we take every element u occurring in $\forall x F$ and the elements $c^{\mathcal{U}}$ for every constant c occurring in $\forall x F$ to be so mentioned), $\mathcal{U} \Vdash F^x_u$, and similarly for the existential quantifier.

It should be clear that W-logic can still be embedded in FOL$^=$ along the lines discussed in section 2; the only change worth mentioning is that $\psi(\forall x F)$ now becomes

$$\forall x \left(\bigwedge_{i=1}^{n} x \neq a_i \wedge \bigwedge_{j=1}^{m} x \neq c_j \rightarrow \psi(F) \right),$$

where the a_i are precisely the parameters and the c_j precisely the constants occurring in $\forall x F$, and similarly for the existential case.

Under the assumption that no two constants assume the same value, FOL$^=$ can also still be embedded in W-logic, essentially along the lines discussed in section 2. Apart from the obvious modification of the quantifier clauses in the definition of the translation φ, we now need to translate not only equations between parameters, but also equations between constants and between parameters and constants. Where a is a parameter, and c and d distinct constants, we let $\varphi(c = d)$ be $\neg(Rcd \rightarrow Rcd)$, $\varphi(c = c)$ be $Pc \rightarrow Pc$, and $\varphi(a = c)$, as well as $\varphi(c = a)$, be $\neg(Rac \rightarrow Rac)$. The proof of the adequacy of φ is then a simplification of that of lemma 4 below.

But as soon as we permit distinct constants to denote the same object, this direction of the translation breaks down, as we can no longer provide an adequate translation for $c = d$:

Let $\mathcal{L}'$ be the language of first-order logic augmented with the two individual constants c and d. Consider the following $\mathcal{L}'$-structures $\mathcal{U}_1$ and $\mathcal{U}_2$. For each $i = 1, 2$, the universe of $\mathcal{U}_i$ is the set ω of natural numbers, the interpretation $R^{\mathcal{U}_i}$ of each n-ary predicate symbol R in $\mathcal{U}_i$ is the full set ω^n (so every predicate applies universally), and the interpretation $c_i := c^{\mathcal{U}_i}$ of the constant c in $\mathcal{U}_i$ is the number 0. The constant d, however, is interpreted in $\mathcal{U}_1$ as $d_1 := d^{\mathcal{U}_1} := 0$, but in $\mathcal{U}_2$ as $d_2 := d^{\mathcal{U}_2} := 1$. We can clearly distinguish between $\mathcal{U}_1$ and $\mathcal{U}_2$ in FOL$^=$ by means of the sentence $c = d$; however, as the following lemma shows, the structures are indistinguishable in W-logic. Hence W-logic is less expressive than FOL$^=$ with respect to languages containing possibly co-denoting individual constants.

Lemma 3. *For every U-sentence A of $\mathcal{L}'$, $\mathcal{U}_1 \Vdash A$ iff $\mathcal{U}_2 \Vdash A$.*

Towards a proof, we first note the following fact. If F is a quantifier-free U-sentence, then for each $i = 1, 2$, if there is *any* function σ, not necessarily 1-1, assigning an element $u \in U$ to each parameter a

occurring in F, such that $\mathcal{U}_i \Vdash F^\sigma$, then we have $\mathcal{U}_i \Vdash F^\sigma$ for *every* such function σ. This is obvious for atomic F, as all predicates are universal in either $\mathcal{U}_i$, and for Boolean combinations it follows easily from the inductive hypothesis.

Next we show that both $\mathcal{U}_i$ permit quantifier elimination in the following sense: For every $\mathcal{L}'$-sentence F there exists a quantifier-free $\mathcal{L}'$-sentence F' in at most the same parameters such that, for any function σ, not necessarily 1-1, assigning elements of U to the parameters in F, and for each $i = 1, 2$: $\mathcal{U}_i \Vdash F^\sigma$ iff $\mathcal{U}_i \Vdash F'^\sigma$.

We prove this claim by induction on the sentence F. In the atomic case, we take F' to be F. Boolean combinations are trivial with the help of the inductive hypothesis. So suppose F is $\forall x G$. Then $\mathcal{U}_i \Vdash F^\sigma$ iff for all $u \in \omega$ other than those mentioned in F^σ, $\mathcal{U}_i \Vdash (G^x_u)^\sigma$. Now let $\sigma[a := u]$ be the function that is just like σ except that it assigns u to the parameter a. Then, if a does not occur in F, we have $\mathcal{U}_i \Vdash (G^x_u)^\sigma$ iff $\mathcal{U}_i \Vdash (G^x_a)^{\sigma[a:=u]}$. Hence, by the inductive hypothesis, $\mathcal{U}_i \Vdash F^\sigma$ iff for all $u \in \omega$ other than those mentioned in F^σ, $\mathcal{U}_i \Vdash [(G^x_a)']^{\sigma[a:=u]}$ (where $(G^x_a)'$ is quantifier-free). But by the fact noted above, this is the case iff $\mathcal{U}_i \Vdash [(G^x_a)']^{\sigma[a:=0]}$ (for the left-to-right direction it is important that there always are elements u other than those mentioned in F, which is the case because the domain is infinite). Now $\mathcal{U}_i \Vdash [(G^x_a)']^{\sigma[a:=0]}$ iff $\mathcal{U}_i \Vdash ([(G^x_a)']^a_c)^\sigma$, and hence we can take F' to be $[(G^x_a)']^a_c$.

Now the lemma follows, because $\mathcal{U}_1$ and $\mathcal{U}_2$ clearly make the same quantifier-free U-sentences W-true.

5. Identity and Co-Denotation

We now show how to overcome the expressive deficit of W-logic with respect to co-denoting individual constants. To this end, we introduce a new binary symbol $\equiv$ into the language of W-logic and extend the definition of a formula by allowing as additional atomic formulas strings of the form $c \equiv d$, where c and d are *individual constants.* Note that the co-denotation symbol must not be flanked, on either side, by a parameter or a variable.

The intended interpretation of $c \equiv d$ is that for some object s in the domain of discourse, c and d both denote s. Co-denotation, so conceived, clearly has a metalinguistic flavor, much like the notion of identity used by Frege in the *Begriffsschrift* [6] (also symbolized by the triple bar). Two points are worth making.

First, there is a certain degree of use-mention confusion in the object language, if the triple bar is interpreted as here suggested. This confusion, however, is harmless, for it is completely resolvable by context: Individual constants occurring to the immediate left or right of a triple bar denote themselves, autonymously; in all other contexts, they denote their usual denotations.

Second, it is not that friends of the identity relation could be prevented from reading formulas of the form $c \equiv d$ as expressing numerical identity between the object denoted by c and d; of course they could. The point is rather that one is not forced to such a reading by the mere desire to restore full expressive power to W-logic.

Let me emphasize again in this connection that quantification into $\equiv$-contexts is syntactically impossible, because bound variables can never occur flanking the triple bar. In this way, we avoid a standard objection against Frege's treatment of identity in the *Begriffsschrift* [6, §8], namely that his view of identity as co-denotation '[if] use and mention are not to be confused, (...) renders a formal treatment of the logic of identity all but impossible' (Church [5, p. 3], see also Furth [7, p. xix]). More precisely, in Richard Mendelsohn's words:

> Frege held that expressions flanking the symbol for identity of content stood for themselves, although in all other contexts they stood for their ordinary contents; combining two such constructions within the scope of a quantifier entailed quantifying into what is essentially an opaque context. [10, p. 60]

By way of example, consider the sentence $\exists x(x \equiv c \wedge P(x))$, where P stands for an ordinary property, say being green. How is our sentence to be interpreted, according to Frege's *Begriffsschrift* theory? Strictly speaking, it should be rendered along these lines: 'Something is such that 'it' co-denotes with 'c' and it is green,' which seems to make little sense. Even without the quotation marks around the pronoun, it doesn't yield the desired reading, for only linguistic items can co-denote with a name, whereas only physical objects can be green.[13]

The point of our prohibition against variables on either side of the triple bar is precisely to eliminate such contexts. Frege was not in a position to make this move; given his semantics for the quantifiers, he had to quantify into $\equiv$-contexts in order to express, for instance, the uniqueness of a relation in §31 of Begriffsschrift (formula115). As I will now show, use of W-logic would have solved this problem for Frege.[14]

It should be clear that W-logic extended by a co-denotation symbol can still be translated into FOL$^=$ simply by rendering $c \equiv d$ as $c = d$. But with the help of the co-denotation symbol, we may now also translate FOL$^=$ into W-logic by revising the definition of the translation function φ given in the previous section to the effect that for distinct constants c and d, $\varphi(c = d)$ now becomes $c \equiv d$.

We can then show the following:

Lemma 4. *Let F be an $\mathcal{L}^=$-sentence containing as constants precisely $c_1, \ldots, c_m$ and as parameters precisely $a_1, \ldots, a_n$. Let $\mathcal{U}$ be a structure and let σ be a* 1-1 *function assigning an element of $U \setminus \{c_1^{\mathcal{U}}, \ldots, c_m^{\mathcal{U}}\}$ to each of the a_i. Then σ satisfies F in $\mathcal{U}$ (in the sense of* FOL$^=$*) iff σ W-satisfies $\varphi(F)$ in $\mathcal{U}$.*

The proof proceeds by induction on the formula F. The atomic cases are all trivial, given that the $\sigma(a_i)$ are all distinct and all distinct from the $c_j^{\mathcal{U}}$. Similarly, Boolean combinations are easily handled by the inductive hypothesis. It remains to verify the quantifier cases; we restrict ourselves to the universal quantifier. So suppose F is of the form $\forall x G$. Then $\varphi(F)$ is

$$\forall x \varphi(G) \wedge \bigwedge_{i=j}^{m} \varphi(G^x_{c_j}) \wedge \bigwedge_{i=1}^{n} \varphi(G^x_{a_i}),$$

where $\varphi(G)$ is $\varphi(G^x_a)^a_x$, a being a fresh parameter.

Now $\mathcal{U} \models \forall x G^\sigma$ iff for every $u \in U$, $\mathcal{U} \models (G^x_u)^\sigma$. The elements of U can be partitioned into three disjoint sets: (A) the values of the constants c_j, (B) the $\sigma(a_i)$, and (C) the rest of U. Hence $\mathcal{U} \models \forall x G^\sigma$ iff all of the following: For each of the constants c_j, $\mathcal{U} \models (G^x_{c_j})^\sigma$, and for each of the parameters a_i, $\mathcal{U} \models (G^x_{a_i})^\sigma$, and for every $u \in U \setminus \{c_1^{\mathcal{U}}, \ldots, c_m^{\mathcal{U}}, \sigma(a_1), \ldots, \sigma(a_n)\}$, $\mathcal{U} \models (G^x_u)^\sigma$. By the induction hypothesis, this is the case iff all of the following: $\mathcal{U} \Vdash \bigwedge_{i=j}^{m} \varphi(G^x_{c_j})^\sigma$, $\mathcal{U} \Vdash \bigwedge_{i=1}^{n} \varphi(G^x_{a_i})^\sigma$, and $\mathcal{U} \Vdash \forall x[\varphi(G)^x_u]^\sigma$. But this in turn is equivalent, by definition of φ, to $\mathcal{U} \Vdash \varphi(\forall x G)^\sigma$.

As a corollary, we obtain the desired result about pure sentences, as (W-)satisfaction by the empty function, which is trivially 1-1 and whose image contains no values of any constants, is just (W-)truth, and lemma 4 is proved.

Notes

[1]See also Hintikka's examples in [8, p. 225].

[2]It is less clear whether the mutual interpretability can serve as a positive argument for the elimination of identity. Since the translation goes both ways, friends of identity might argue that FOL$^=$ simply makes a commitment to identity explicit that W-logic hides in its quantifier semantics.

[3]We are here, for the most part, following Smullyan [15, chapter 4]. Note that Smullyan's parameters are essentially typographically distinguished free variables, as used e.g. in the Gentzen tradition (see for example Schütte [12]). The use of

such parameters is convenient for tableaux calculi, but by no means indispensible. W-logic can just as well be done with just one sort of individual variable; we will indicate the adjustments necessary to the translations between W-logic and FOL$^=$ in the notes. Similarly, the use of elements of a domain as constituents of formulas is simply a matter of convenience that allows one to suppress mention of variable assignments in the definition of truth in a structure; the influential textbook by Shoenfield [13] has made this technique popular.

[4]The language $\mathcal{L}^=$ of FOL$^=$ results from $\mathcal{L}$ by the addition of a distinguished binary predicate symbol $=$, the equality symbol; semantically, it is interpreted as true identity.

[5]As promised, we briefly address languages without a set of designated free variables (parameters) in the endnotes. The translation ψ is actually simpler when working in such a language. The clauses for atomic formulae and Boolean connectives are essentially the same; however, in the universal quantifier case, $\psi(\forall xF)$ is literally $\forall x\,(\bigwedge_{i=1}^{n} x \neq a_i \to \psi(F))$, without needing to introduce $\psi(F)$ as an abbreviation for $\psi(F^x_a)^a_x$.

[6]Again, here are the relevant translation clauses with respect to first-order languages that don't have typographically distinguished free variables (parameters):

(1) $\varphi(x = x)$ is $Px \to Px$
(2) $\varphi(x = y)$ is $\neg(Rxy \to Rxy)$, where x and y are distinct individual variables.
(3) $\varphi(\forall xF)$ is

$$\forall x\varphi(F) \wedge \bigwedge_{i=1}^{n} \varphi(F^x_{x_i}),$$

where it is assumed that, before substituting x_i for x in F, bound variables in F have been renamed so as to avoid unintentional capturing of x_i in the result of the substitution (and similarly for the existential case).

Note how, in the quantifier case, our original version *with* parameters obviates the need for renaming bound variables in the second conjunct. On the other hand, languages with parameters are clumsier with respect to the first conjunct, where one must first replace the bound variable x with a fresh parameter a, then effect the translation, and then re-substitute x for a in the result.

[7]These tableaux rules are the obvious analogues of the rules for the Gentzen sequent calculus introduced in Wehmeier [16]. Soundness and completeness for the tableaux calculus may thus be inferred from the corresponding properties of that sequent calculus. We wish to note, however, that the soundness and completeness of the sequent calculus were established only indirectly in the mentioned paper, that is, by reference to a sound and complete calculus for FOL$^=$. The proofs outlined here connect the W-procedure directly with the semantics for W-logic and should thus be of independent interest. For the record, we would like to note that soundness and completeness proofs for what is essentially the W-procedure were first given in an unpublished term paper by Sam Hillier, written under my supervision.

[8]Let us note here that (W-UI) must insist on the instantiating parameter already occurring on the branch (except in the case where there are no parameters on the branch whatsoever). In tableaux calculi for FOL, it is strategically a good idea to use such instantiating parameters, but certainly not required for the soundness of the calculus. The W-procedure, however, actually becomes unsound if arbitrary

instantiating parameters are allowed (even if these are restricted to those not ocurring in the quantified sentence itself). This is due to the fact that it is impossible to properly extend σ in a 1-1 way if the range of σ already exhausts U. Here's an example:

$$\frac{(1)\quad \forall x \forall y(Rxy \wedge \neg Rxy)}{(2)\quad \forall y(Ray \wedge \neg Ray)}\ \text{UI from (1)}$$

At this point, the tree cannot be continued in accordance with the W-procedure: Instantiating the sentence in line (2) to the parameter a is not permissible, because a occurs in that very sentence. Instantiating to another parameter is not allowed, because there are already parameters on the branch, viz. a. And this is as it should be, because the sentence at the origin is W-valid in every domain of cardinality 1. If we allowed instantiation to arbitrary parameters not occurring in the quantified sentence under consideration, we'd be able to continue the above tree (obtaining line (3) from line (2) by an illegitimate application of universal instantiation) as follows

$$\cfrac{(3)\quad Rab \wedge \neg Rab}{\cfrac{(4)\quad Rab}{(5)\quad \neg Rab}\ \wedge\text{-rule from (3)}}\ \wedge\text{-rule from (3)}$$

and end up with a closed tree, incorrectly indicating that the sentence in line (1) should be W-unsatisfiable.

[9]Beginning with a single-node tree T_0 and successively applying the rules of the W-procedure, one obtains a sequence $T_0, T_1, T_2, \ldots$ of trees. If every T_i is extendable by a rule application, this sequence may be continued indefinitely. In this case, we will also refer to its limit $\bigcup_{i=0}^{\infty} T_i$ as a tree constructed according to the W-procedure.

[10]It is perhaps interesting to note that, by imposing *restrictions* on the quantifier rules of the Burgess-Boolos calculus for FOL, one obtains a calculus sound and complete for W-logic, which we have shown to be stronger than FOL, viz. equivalent to FOL$^=$. Hintikka makes a similar observation with respect to his calculus in [8, p. 237].

[11]Indeed, it will produce a finite model of minimal cardinality. Proof: let the model $\mathcal{U}$ of the argument just given be of minimal cardinality. This was observed, for the Burgess-Boolos calculus, by Perry Smith [14].

[12]Parameters are not individual constants in the sense considered here, but rather typographically distinguished free variables. Constants, as explained below, are interpreted as fixed objects in a structure, whereas parameters may assume, in the same model, arbitrary values over the domain of discourse.

[13]See also Charles Caton's discussion in [4, pp. 174-175].

[14]Mendelsohn [10, pp. 60-61] also points out that neither the autonymous use of individual constants in $\equiv$-contexts nor the simultaneous autonymous and non-autonymous use of constants in separate contexts constitutes a logical error. It is only the simultaneous quantification into a position reserved for autonymous terms and a position reserved for non-autonymous terms that arguably creates gibberish.

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