Russell's Paradox in Consistent Fragments of Frege's *Grundgesetze der Arithmetik*

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Abstract

We provide an overview of consistent fragments of the theory of Frege's *Grundgesetze der Arithmetik* that arise by restricting the second-order comprehension schema. We discuss how such theories avoid inconsistency and show how the reasoning underlying Russell's paradox can be put to use in an investigation of these fragments.

1 Introduction.

On June 16th, 1902, Russell communicates 'his' paradox to Frege:

> Let w be the predicate of being a predicate that cannot be predicated of itself. Can one predicate w of itself? From any answer follows its opposite. Therefore one must infer that w is not a predicate. Likewise, there is no class (as a whole) of those classes that as wholes do not belong to themselves. From this I infer that under certain circumstances a definable set does not form a whole.[1]

[1] Cf. [10, p. 211] (the English translation is mine): 'Sei w das Prädicat, ein Prädicat zu sein welches von sich selbst nicht prädicirt werden kann. Kann man w von sich selbst prädiciren? Aus jeder Antwort folgt das Gegentheil. Deshalb muss man schliessen dass w kein Prädicat ist. Ebenso giebt es keine Klasse (als Ganzes) derjenigen Klassen die als Ganze sich selber nicht angehören. Daraus schliesse ich dass unter gewissen Umständen eine definierbare Menge kein Ganzes bildet.'

In a postscript to this letter, Russell uses Peano's notation to give the first symbolic expression of the paradox:

$$w = \mathrm{cls} \cap x \ni (x \sim \epsilon\, x). \supset: w \,\epsilon\, w. = .w \sim \epsilon\, w.^{2}$$

One would imagine that Frege was not much impressed by the predicability version of the antinomy that Russell mentions first (which is basically the heterologicality paradox), as it is obviated, within the *begriffsschrift*, by the stratification of functions into levels. The class version of the contradiction, however, nowadays known as Russell's paradox, Frege immediately recognised as being reconstructible within the theory of *Grundgesetze*. He regarded the occurrence of this contradiction as a serious blow to his life's work – one can clearly sense his horror between the lines of his answer to Russell, written on June 22nd, 1902:

> Your discovery of the contradiction utterly surprised and, I am almost inclined to say, dismayed me, as the foundation upon which I believed arithmetic to be built, thereby begins to rock. It thus seems that the transformation of the generality of an identity into an identity of value-ranges (§9 of my Grundgesetze) is not always permitted, that my law V (§20 p. 36) is false and that my explanations in §31 do not suffice to secure my combinations of signs a denotation in all cases.[3]

It is clear from these remarks that Frege himself considered his *Grundgesetz* V as the source of the antinomy. This is a very reasonable diagnosis, because basic law V figures prominently in the derivation of the contradiction, as we shall see shortly. Most philosophers have followed Frege in this assessment. Michael Dummett has pointed out, however, that another contentious principle plays an equally prominent role in the genesis of the contradiction, viz., Frege's substitution rule for 'free' second-order variables [8, §48, subsection 9, pp. 62-63]. Such a rule is equivalent, under certain conditions, to the full (impredicative) second-order comprehension schema, and it is this impredicativity that Dummett takes to be responsible for the contradiction.[4]

[2]See [10, p. 212]. In modern notation, Russell's formula reads as follows: $w = \{x : \neg x \in x\} \rightarrow (w \in w \leftrightarrow \neg w \in w)$.

[3]Cf. [10, p. 213] (the English translation is mine): 'Ihre Entdeckung des Widerspruchs hat mich auf's Höchste überrascht und, fast möchte ich sagen, bestürzt, weil dadurch der Grund, auf dem ich die Arithmetik sich aufzubauen dachte, in's Wanken geräth. Es scheint danach, dass die Umwandlung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit (§9 meiner Grundgesetze) nicht immer erlaubt ist, dass mein Gesetz V (§20 S. 36) falsch ist und dass meine Ausführungen im §31 nicht genügen, in allen Fällen meinen Zeichenverbindungen eine Bedeutung zu sichern.'

[4]Cf. [6, chapter 17] and, for critical discussion, [3].

The bulk of this paper will be concerned with the problem of weakening the comprehension schema in such a way as to obtain a consistent fragment of Frege's original theory[5].

To set the stage, let us begin by considering the version of Russell's paradox that Frege discusses first in the *Nachwort* to the second volume of *Grundgesetze* [9, pp. 256-257] (as reconstructed in terms of axiomatic second-order logic). Recall that, according to Frege, there is, associated with any second-order entity (concept) F, a certain first-order entity (object), the concept's extension (course-of values, value-range) $(\hat{x})F(x)$. These extensions are governed by *Grundgesetz* V:

$$\forall F \forall G\left((\hat{x})F(x) = (\hat{y})G(y) \leftrightarrow \forall z(F(z) \leftrightarrow G(z))\right).$$

That is, for any concepts F and G, their extensions $(\hat{x})F(x)$ and $(\hat{y})G(y)$ coincide if and only if the same objects fall under F as fall under G. Thus, basic law V is essentially an extensionality axiom for value ranges.

Russell's paradox now arises as follows: Consider the concept R such that an object a falls under R if and only if a is the value-range of some concept F under which a does not fall. By (impredicative) second-order comprehension, such a concept R exists:

$$\exists R \forall a\left(R(a) \leftrightarrow \exists F(a = (\hat{z})F(z) \wedge \neg F(a))\right).$$

Now let r be the extension of R, $r = (\hat{y})R(y)$. Suppose that r falls under R. By definition of R, r is then the extension of some F such that r does not fall under F, i.e. $\exists F(r = (\hat{z})F(z) \wedge \neg F(r))$. So $(\hat{z})F(z) = r = (\hat{y})R(y)$. Using basic law V, we find that $\forall x(F(x) \leftrightarrow R(x))$, that is, F and R are coextensional. Hence, since r does not fall under F, we also have $\neg R(r)$. Cancelling the assumption that r falls under R, we have shown that $R(r) \rightarrow \neg R(r)$, hence $\neg R(r)$. By definition of R, this means that for any concept F, if r is the extension of F, then r falls under F, i.e., $\forall F(r = (\hat{z})F(z) \rightarrow F(r))$. By specialising the universally quantified second-order variable F to R (R is in the second-order domain of quantification by the above instance of the comprehension schema), and noting that r is the extension of R, we obtain $R(r)$. So $\neg R(r)$ and $R(r)$ – voilà the contradiction.

Let us now introduce a symbol for the membership relation by the following definition:

$$x \in y \leftrightarrow \forall F(y = (\hat{z})F(z) \rightarrow F(x)),$$

[5]We shall, in fact, not work with the term logic of Frege's *original* theory, but rather with a reconstruction of this theory in terms of second-order predicate logics. This deviation from Frege's setting would seem to be harmless as regards the source of the Russell antinomy.

that is, x is a member of y if and only if, whenever y is the extension of some concept F, x falls under F. We can then immediately obtain the better known class version of Russell's paradox from the derivation above: Russell's condition $x \notin x$ transforms, upon elimination of the defined notion, into $\exists F(x = (\hat{z})F(z) \wedge \neg F(x))$, that is, into x's falling under the concept R introduced above, and the Russell class $\{y : y \notin y\}$ is just r, the extension of R.[6]

As is well known, Russell's paradox is closely related to Cantor's theorem. In fact, Russell himself discovered his paradox through an analysis of Cantor's theorem.[7] Cantor's theorem says that there is no bijection between the first- and the second-order entities (or, in more familiar terms, between the members of a given set A and the subsets of A). The usual proof of this result shows, more specifically, that there is no map from the first- *onto* the second-order universe. Alternatively, in order to prove Cantor's theorem, one can show that there is no *1-1* map from the second- into the first-order universe. It is this alternative proof that reveals the connection to Russell's paradox: Suppose that $G \mapsto G^*$ is a 1-1 function from the second- into the first-order domain. Now consider the concept C with the following comprehension property:

$$C(x) \leftrightarrow \exists F(x = F^* \wedge \neg F(x)).$$

C^* is a first-order entity, so we may ask whether C^* falls under C. Suppose it does. Then by definition of C, there is a concept F such that $C^* = F^*$ and $\neg F(C^*)$. As $G \mapsto G^*$ is 1-1, F and C are the same, and so $\neg C(C^*)$. Cancelling the supposition that $C(C^*)$, we obtain $C(C^*) \rightarrow \neg C(C^*)$, i.e. $\neg C(C^*)$. Again by definition of C, this means that $\forall F(C^* = F^* \rightarrow F(C^*))$, from which $C(C^*)$ immediately follows. As $C(C^*)$ and $\neg C(C^*)$ cannot both be true, we must conclude that $G \mapsto G^*$ is not 1-1.

It should be clear that, in this version, Cantor's theorem basically *is* Russell's paradox: Reading $(\hat{x})G(x)$ instead of G^* and noting that basic law V claims this function to be 1-1, the proofs are literally identical.[8] It is the more surprising that Frege failed to realise at the outset that his value-range

[6] Frege himself discusses (a variant of) this version of the paradox in the *Nachwort* [9, p. 257], making use of his analogue of the membership relation introduced in §34 (which is in fact, due to the peculiar nature of Frege's system, an application function). We speak of a 'variant' here because Frege defines the membership relation $x \in y$ as $\exists F(y = (\hat{z})F(z) \wedge F(x))$ (this definition is slightly more natural than the one used above: if y is not a value-range, then, according to the latter definition, it has no members, whereas according to the one we have given, everything will then be a member of y).

[7] Cf. Russell's letter to Frege of June 24th, 1902 [10, pp. 215-6].

[8] Cf. [4] for an illuminating discussion of these matters.

function would lead to inconsistency, as he was clearly aware of Cantor's theorem: In §164 of the second volume of *Grundgesetze*, he mentions that there are more classes of natural numbers than natural numbers, without crediting this insight to Cantor:

> Now an infinite number, which we have called Endless, belongs to the concept *finite number*; but this infinity does not yet suffice. If we call the extension of a concept that is subordinate to the concept *finite number* a *class of finite numbers*, then an infinite number that is greater than Endless belongs to the concept *class of finite numbers*; i.e. the concept *finite number* can be mapped into the concept *class of finite numbers*, but not vice versa the latter into the former.[9]

It remains puzzling why Frege did not see the consequences of Cantor's result for his own logical system. In any case, Russell did, and we shall now discuss how to avoid the antinomy by weakening the second-order comprehension schema.

2 Restricting the Comprehension Schema

We saw in the informal derivation of Russell's antinomy above that a certain instance of the comprehension schema plays a pivotal role in the proof, viz., $\exists R \forall x\, (R(x) \leftrightarrow \exists F(x = (\hat{z})F(z) \wedge \neg F(x)))$. It seems natural to ask what happens when comprehension is not available, or available only for formulae of a complexity below that of $\exists F(x = (\hat{z})F(z) \wedge \neg F(x))$. Peter Schroeder-Heister [14] conjectured that, in the complete absence of comprehension, that is, in the first-order fragment of Frege's theory, the antinomy would no longer be derivable. This was later confirmed by Terence Parsons [13]. We

[9]Cf. [9, p. 161] (the translation is mine): 'Nun kommt ja dem Begriffe *endliche Anzahl* eine unendliche Anzahl zu, die wir Endlos genannt haben; aber diese Unendlichkeit genügt noch nicht. Nennen wir den Umfang eines Begriffes, der dem Begriffe *endliche Anzahl* untergeordnet ist, eine *Klasse endlicher Anzahlen*, so kommt dem Begriffe *Klasse endlicher Anzahlen* eine unendliche Anzahl zu, die grösser als Endlos ist; d.h. es lässt sich der Begriff *endliche Anzahl* abbilden in den Begriff *Klasse endlicher Anzahlen*, aber nicht umgekehrt dieser in jenen.' It seems worth noting that Frege here deviates from the terminology he introduced in volume I: We can clearly map the sets of natural numbers into the natural numbers – by mapping every set to 0, say. But there is no 1-1 map from the sets of numbers into the numbers, so Frege presumably means 'map 1-1 into' when he says 'map into'. Or perhaps the vice versa clause is intended to mean that, whenever a relation maps the natural numbers into the sets of natural numbers, the converse of that relation will not map the sets into the numbers. The point seems marginal, but Frege's sloppiness is somewhat surprising.

shall consider this system in subsection 2.1. Parsons' consistency proof was then extended, by Richard Heck [12], to the predicative fragment of Frege's theory, where the comprehension formulae must not contain any second-order quantifiers. We shall discuss Heck's theory, and some observations concerning it made in [15], in subsection 2.2. Subsequently, we briefly turn to a discussion of the Δ^1_1-CA fragment of Frege's theory that was recently proven consistent by Fernando Ferreira and the present author [7]. In the final subsection 2.4, we consider a related, though weaker theory proved consistent and discussed in [15], and contrast its linguistic setup with that of the theories discussed earlier.

2.1 The First-Order Fragment

We shall follow Heck [12] rather than Parsons [13] in setting out the first-order fragment. This is for reasons of convenience, as Parsons conscientiously works with a Fregean term logic, whereas the systems to be considered later are all based on predicate logic. In a first-order theory, we can obviously not formulate basic law V with the help of second-order universal quantification, as we did above. Thus we consider here *schema V* instead, that is, all instances of

$$(\hat{x})\phi(x) = (\hat{y})\psi(y) \leftrightarrow \forall z(\phi(z) \leftrightarrow \psi(z)),$$

where ϕ and ψ are any formulae of the first-order fragment's language L_1. The terms and formulae of L_1 are generated by the following inductive definition:

1. Every individual variable is a term.

2. If s and t are terms, then $s = t$ is a formula.

3. Boolean combinations of formulae are formulae.

4. If x is an individual variable and $\phi(x)$ is a formula, then $\forall x\phi(x)$ is a formula and $(\hat{x})\phi(x)$ is a term (a value-range or VR term, as we shall say).

Parsons builds a structure for this language satisfying all instances of schema V as follows. Take ω as the domain of quantification. Assign the closed VR terms (where parameters from ω are allowed) natural numbers as ranks according to the rule: If $\phi(x)$ contains no VR terms, then the rank of $(\hat{x})\phi(x)$ is 0; if it does contain VR terms, and the maximal rank of a VR term occurring in it is n, then the rank of $(\hat{x})\phi(x)$ is $n + 1$. Within a given rank, order the closed VR terms arbitrarily into an ω-sequence. Partition ω (or some infinite subset of it) into infinitely many infinite *candidate sets* U_i $(i < \omega)$. Now assign values from $\bigcup_{i=0}^{\infty} U_i$ to the closed VR terms by recursion on ω^2. Suppose you are treating the mth VR term of rank n, $(\hat{x})\phi(x)$, say. All closed instances of VR terms occurring in $\phi(x)$ have rank lower than n and have thus already been assigned values. It is therefore determined whether for some VR term $(\hat{x})\psi(x)$ treated at some earlier stage we have $\forall x(\phi(x) \leftrightarrow \psi(x))$. If this is the case, we assign to $(\hat{x})\phi(x)$ whatever was assigned to $(\hat{x})\psi(x)$ before. Otherwise, we assign it the first element of U_n not yet used. In this way, we clearly obtain a model of the full schema V.

We note in passing that John Burgess [5] has recently given a more constructive proof of Parsons' result. Furthermore, Warren Goldfarb [11] has shown that the first-order fragment is recursively undecidable, so it is not completely trivial mathematically (although it is unknown whether the theory is *essentially* undecidable). However, it does not seem likely that any interesting mathematics can be developed within this fragment.

2.2 The Predicative Fragment

The language L_2 of Heck's predicative fragment **H** results from Parsons' L_1 by adding second-order variables and quantifiers. That is, the terms and formulae of L_2 are generated by the clauses given above for L_1, supplemented by a clause for building atomic formulae $F(t)$ from second-order variables F and terms t, and closure under universal second-order quantification. Schema V is as before, where of course the instances are now formed by inserting arbitrary L_2-formulae for ϕ and ψ. In addition to schema V, **H** has as axioms all instances of predicative comprehension, that is, all instances of

$$\exists F \forall x(F(x) \leftrightarrow \phi(x)),$$

where ϕ contains no second-order quantifier. The deductive apparatus of **H** may be taken to be that of two-sorted first-order logic, where the objects and concepts represent the sorts.

We start by building a model for the first-order fragment of **H** by Parsons' procedure, assigning values to closed VR terms containing no second-order variables (but possibly parameters from ω), where we take care to choose the U_i in such a way that $\bigcup_{i=0}^{\infty} U_i$ has an infinite complement. We expand the resulting first-order structure to a second-order structure by letting the second-order quantifier range over the first-order definable subsets of ω. Now we assign values to those closed VR terms of L_2 that contain second-order parameters, but no second-order quantifiers, simply by expanding the parameters to their first-order definitions and choosing the value that was assigned to the corresponding first-order VR term. At this stage, we have stipulated enough in order for the predicative comprehension schema to be valid, as is easy to see. It remains to take care of those VR terms that contain, besides possibly first- and second-order parameters, second-order quantifiers. This can be done more or less as in the original Parsons procedure, using the infinite complement of $\bigcup_{i=0}^{\infty} U_i$ as values for the properly impredicative VR terms. It is then easy to check that all instances of schema V hold.

As was observed in [15], the theory **H** has a curious property: it proves the existence of objects that are not value-ranges. That is, the sentence $\exists x \forall G\, (x \neq (\hat{y})G(y))$ is a theorem of **H**. There is even a term of L_2 witnessing this existential sentence, viz. the term r for the Russell class introduced above: $(\hat{z})(\exists F(z = (\hat{v})F(v) \wedge \neg F(z)))$. In other words, there is a value-range term of L_2 of which **H** proves that it does not *denote* a value-range. The proof is simple, and makes no use of the comprehension schema at all:

Argue in **H**. As before, let $R(x)$ be the formula $\exists F(x = (\hat{v})F(v) \wedge \neg F(x))$. Now assume $R(r)$, that is, $\exists F(r = (\hat{v})F(v) \wedge \neg F(r))$. Take such an F. Since $(\hat{v})F(v) = r = (\hat{z})R(z)$, by the appropriate instance of schema V, $\forall x(F(x) \leftrightarrow R(x))$. So, since $\neg F(r)$, we also have $\neg R(r)$. As above, we cancel the assumption and conclude $R(r) \rightarrow \neg R(r)$, i.e. $\neg R(r)$. This means $\forall F(r = (\hat{v})F(v) \rightarrow F(r))$. Now suppose that for some second-order G, $r = (\hat{y})G(y)$. It follows that $G(r)$ holds; but by schema V, it also follows that $\forall x(R(x) \leftrightarrow G(x))$, and hence $R(r)$, contradiction. So the assumption must be false, and we have proven $\forall G\, (r \neq (\hat{y})G(y))$. By exploiting other Russellian antinomies, one can indeed provide any finite number of terms all of which fail to denote value-ranges, see [15].

We shall come back to these matters in section 2.4. For now we note that, as Heck [12] has shown, Robinson's arithmetic **Q** can be interpreted in **H**, and so the theory is essentially undecidable. But it is unclear, though unlikely, whether much more arithmetic can be developed within the predicative fragment. We now turn briefly to an extension of Heck's predicative fragment whose consistency remained open in [12], viz. the Δ^1_1-CA fragment.

2.3 The Δ^1_1-CA Fragment

Inspection of the proof of the Russell antinomy shows that the instance of comprehension used has complexity Σ^1_1, i.e. the comprehension formula has the form $\exists F\phi(F)$, where ϕ is predicative. Alternatively, one could use a comprehension formula of complexity Π^1_1 (that is, of the form $\forall F\phi(F)$ with predicative ϕ), viz. the variant $\forall F(x = (\hat{z})F(z) \rightarrow \neg F(x))$ of the Russell concept to generate the contradiction. Hence both Σ^1_1- and Π^1_1-comprehension are inconsistent with schema V (in fact, either one of these schemas implies the other). This led Heck [12] to ask whether the schema of Δ^1_1-comprehension would be inconsistent with basic law V as well, where this schema can be described as the set of all instances of

$$\forall x(\phi(x) \leftrightarrow \psi(x)) \rightarrow \exists F \forall x(F(x) \leftrightarrow \phi(x)),$$

ϕ being a Σ^1_1-formula, ψ aΠ^1_1-formula, and F not occurring free in ϕ. A partial answer was given in [15], to which we shall come back in the next subsection; however, the linguistic setting of the theory considered in that paper is rather different from that of **H**. In [7] it was finally shown that Δ^1_1-comprehension can indeed be consistently added to **H**. The idea of the proof is as follows: We start by building a model of the first-order fragment in the spirit of Parsons. This model has a recursively saturated elementary extension, on which we then perform the Heck construction to obtain a model of **H**. It can then be shown, by methods introduced in [1], that the recursive saturation of the first-order part forces this model to validate even Δ^1_1-comprehension.

Adding Δ^1_1-comprehension to **H** obviously results in a stronger theory; it remains doubtful, however, whether much more arithmetic can be done within this extension than within the original theory. Clearly, the observations concerning the existence of objects that are not value-ranges continue to hold for this extension of **H**.

2.4 The Theory T_Δ

To motivate the discussion in this section, let us return to the curious feature of **H** (and any consistent extension thereof) noted in subsection 2.2 above, viz. the provable existence of objects other than value-ranges. As pointed out, **H** proves $\forall F\,(r \neq (\hat{z})F(z))$ for a term r that purports to denote a value range. It cannot denote the value range of any *concept*, however, because the formula from which it is derived is properly impredicative, so that no second-order entity corresponds to it. Only a VR term derived from a predicative formula will actually denote the value-range of some second-order entity. So we have many mock VR terms around – VR terms constructed out of impredicative formulae that do not define an entity in the second-order domain. As Heck [12] points out, such impredicative VR terms are mock also with regard to the membership relation: If we let $x \in y$ stand for $\exists F(y = (\hat{z})F(z) \wedge F(x))$, we will be able to prove, within **H**, that $x \in (\hat{z})\phi(z)$ always implies $\phi(x)$; however, only for predicative ϕ does **H** prove the converse.[10]

[10] On the alternative definition of membership $x \in y :\equiv \forall F(y = (\hat{z})F(z) \rightarrow F(x))$, the converse will always be provable, but the original direction only for predicative ϕ.

As shown in [15], the provable existence of non-value-ranges does not, as one might expect, hinge on the existence of mock VR terms. The theory T_Δ proved consistent there is based on a linguistic setting rather different from those of the other fragments considered so far. This is because the value range operator is construed as a third-order function symbol, which accordingly can only be attached to second-order variables, yielding terms of the form $\hat{F}$, but not to arbitrary formulae. *Grundgesetz* V is formulated as one single second-order axiom[11]:

$$\forall F \forall G \left(\hat{F} = \hat{G} \leftrightarrow \forall z (F(z) \leftrightarrow G(z)) \right).$$

In addition, T_Δ has all instances of the Δ^1_1 comprehension schema as axioms. While we may of course introduce VR terms $(\hat{z})\phi(z)$ into this language by way of definitional extension, this will be possible only for formulae ϕ that are provably Δ^1_1, and so all VR terms introducible in T_Δ do actually denote value-ranges of second-order entities. Nevertheless, we can still prove that some things are not value-ranges; indeed, the non-value-ranges do not even form a concept. This can be seen as follows:

Argue in T_Δ and suppose $\exists H \, \forall x (H(x) \leftrightarrow \exists F (x = \hat{F}))$. Then, essentially, the Σ^1_1- and the Π^1_1-version of the Russell formula are equivalent, and hence Δ^1_1: $\exists F (x = \hat{F} \wedge \neg F(x)) \leftrightarrow \forall G (Hx \wedge (x = \hat{G} \rightarrow \neg G(x)))$. Thus, a second-order entity is defined by the Russell formula, and the antinomy follows as before. So there is no concept under which precisely the value-ranges fall. It follows that there must be objects other than value-ranges (otherwise the value-range concept would simply be the universal concept, which is predicatively defined by the formula $x = x$).

3 Closing Remarks

We have surveyed a number of consistent fragments of (a second-order predicate logic reconstruction of) the theory of Frege's *Grundgesetze der Arithmetik* that arise by restricting the second-order comprehension schema. Presumably, none of these theories are of sufficient mathematical strength to provide a reconstruction of arithmetic, or even real analysis, in a Fregean spirit. But even if it turned out that a significant portion of arithmetic could be so developed, the fact remains that these theories (except the first-order fragment) prove certain claims that would seem to be unacceptable for a logicist, in particular, the existence of objects that are not value-ranges. The

[11]Note that, in the context of **H** and its consistent extensions, this axiom trivially follows from schema V, but not vice versa.

point is not that these theories prove the existence of infinitely many objects *simpliciter*, for, if they were to provide a foundation for arithmetic, they had better do so. But the existence of *urelemente* – in fact arbitrarily (finitely) many of them – should certainly not follow from a theory that deserves the epithet 'logical'. As we cannot go beyond Δ^1_1-comprehension without generating an inconsistency, so that the fragment of section 2.3 is the strongest consistent theory obtainable by restricting the comprehension schema, it seems that this strategy does not hold much promise for a logicist foundation of mathematics. The situation may be otherwise with alternative directions of restricting or modifying Frege's system, e.g. modifications of *Grundgesetz* V (retaining full comprehension), as discussed for instance in [2].

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